Resonant Deloc. on the Complete Graph

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Based on:

M.A. - S. Warzel: "On the ubiquity of the Cauchy distribution in spectral problems" ('13)

M.A. - M. Shamis - S. Warzel: "Partial delocalization on the complete graph" (2014)

Eigenfunction hybridization (tunnelling amplitude vs. energy gaps)

Reminder from QM 101: Two-level system $H = \begin{pmatrix} E_1 & \tau \\ \tau^* & E_2 \end{pmatrix}$

Energy gap: $\Delta E := E_1 - E_2$ **Tunneling amplitude:** τ.

- Case $|\Delta E| \gg |\tau|$: **Localization**

$$\psi_1 \approx (1, 0) , \qquad \psi_2 \approx (0, 1) .$$

- Case $|\Delta E| \ll |\tau|$: **Hybridized eigenfunctions**

$$\psi_1 \approx \frac{1}{\sqrt{2}} (1, 1) , \qquad \psi_2 \approx \frac{1}{\sqrt{2}} (1, -1) .$$

Quasimodes & their tunnelling amplitude

Definition:

1. A **quasi-mode** (qm) with discrepancy d for a self-adjoint operator H is a pair (E, ψ) s.t.
$$\|(H - E)\psi\| \leq d\|\psi\| \, .$$
2. The pairwise **tunnelling amplitude**, among orthogonal qm's of energy close to E may be defined as $\boxed{\tau_{jk}(E)}$ in
$$P_{jk}(H - E)^{-1}P_{jk} = \begin{bmatrix} e_j + \sigma_{jj}(E) & \tau_{jk}(E) \\ \tau_{kj}(E) & e_k + \sigma_{kk}(E) \end{bmatrix}^{-1} \, .$$
(the "Schur complement" representation).

Seems reasonable to expect:

If the typical **gap size** for quasi-modes is $\Delta(E)$, the condition for **resonant delocalization** at energies $E + \Theta(\Delta E)$ is:

$$\boxed{\Delta(E) \leq |\tau_{jk}(E)| \, .}$$

Question: how to deal with many co-resonating modes?

Example: Schrödinger operator on the complete graph (of M sites)

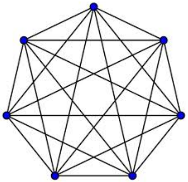
$$\boxed{H_M = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M \; V}$$

with:

- $\langle\varphi_0| = (1, 1, \ldots, 1)/\sqrt{M}$,
- $V_1, V_2, \ldots V_M$ iid standard Gaussian rv's, i.e.

$$\varrho(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2},$$

- $\kappa_M := \lambda / \sqrt{2 \log M}$.



Remarks:

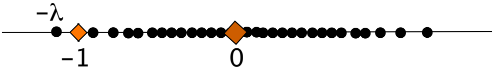
- Choice of (κ_M) motivated by: $\boxed{\max\{V_1, ..., V_M\} \overset{inProb}{=} \sqrt{2\log M} + o(1)\,.}$
- The spectrum of H for $\boxed{M \to \infty}$:

$$\sigma(H_M) \longrightarrow [-\lambda, \lambda] \cup \{-1, 0\} \quad \text{(on the 'macroscopic scale')}\,.$$

- Eigenvalues interlace with the values of $K_M V$
- Studied earlier by Bogachev and Molchanov ('89), and Ossipov ('13) - both works focused on localization.

Two phase transitions for $H_M = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M \; V$

Quasi-modes: $|\varphi_0\rangle$ (extended), and $|\delta_j\rangle$ $j = 1, ..., M$ (localized).



1. A transition at the spectral edge (1^{st}-order), at $\boxed{\lambda = 1}$:

$\lambda < 1 :$ $E_0 = -1 + o(1)\,,$ $\Psi_0 \approx \varphi_0$ (the ground state is extended)

$\lambda > 1 :$ $E_0 = -\lambda + o(1)\,,$ $\Psi_0 \approx \delta_{argmin(V)}$ (the ground state is localized except for 'avoided crossings')

2. Emergence of a band of semi-delocalized states:

at energies near $E = -1$, for $\boxed{\lambda > \sqrt{2}}$.

A similar band is found also near $E = 0$ for all $\lambda > 0$.

The characteristic equation

Proposition
The eigenvalues of H_M intertwine with the values of κV.
The spectrum of H_M consists of the collection of energies E for which

$$F_M(E) \ := \ \frac{1}{M} \sum_{x=1}^{M} \frac{1}{\kappa_M V(x) - E} \ = \ 1 \,, \tag{1}$$

and the corresponding eigenfunctions are given by:

$$\psi_E(x) = \frac{Const.}{\kappa_M V(x) - E} \,. \tag{2}$$

Furthermore: The eigenvalues of H_M intertwine with the values of κV.

Proof: allows to deduce, by standard arguments, that for any $z \in \mathbb{C} \backslash \mathbb{R}$:

$$\frac{1}{H_M - z} \ = \frac{1}{\kappa_M V - z} \ + [1 - F_M(z)]^{-1} \ \frac{1}{\kappa_M V - z} \ |\varphi_0\rangle\langle\varphi_0| \ \frac{1}{\kappa_M V - z} \tag{3}$$

and, in particular, $\langle \varphi_0 \,, \, (H_M - z)^{-1} \varphi_0 \rangle \ = \ (F_M(z)^{-1} - 1)^{-1}$. The spectrum and eigenfunctions of H_M are then read from the poles and residues of its resolvent. □

The scaling limit

Zooming onto scaling windows centered at a sequence of energies $\mathcal{E}_M$ with:

$$\lim_{M\to\infty} \mathcal{E}_M = \mathcal{E} \in [-\lambda, \lambda], \quad \text{and} \qquad |\mathcal{E}_M - \mathcal{E}| \le C/\ln M \, ,$$

denote

$$\boxed{u_{n,M} := \frac{E_{n,M} - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}}\,, \qquad \boxed{\omega_{n,M} := \frac{\kappa_M V_j - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}\,.}$$

rescaled eigenvalues — rescaled potential values

Questions of interest:

1. the nature of the limiting point process of the rescaled eigenvalues (including: extent of level repulsion (?), and relation to rescaled potential values)
2. the nature of the corresponding eigenfunctions (extended versus localized, and possible meaning of these terms).

Results (informal summary)

Theorem 1 [Bands of partial delocalization (A., Shamis, Warzel)]
I. If either

- $\mathcal{E} = 0$, $\lambda > 0$;   or
- $\mathcal{E} = -1$, and $\lambda > \sqrt{2}$,   ($\searrow \varrho$'s Hilbert transform)

and additionally the lim exists: $\lim_{M\to\infty} M\Delta_M(\mathcal{E}) \left(1 - \kappa_M^{-1}\,\overline{\varrho}\,(\mathcal{E}_M/\kappa_M)\right) =: \alpha$
then the rescaled eigenvalue point process converges in distribution to the Šeba point process at level α [defined below].

II. the eigenvalues within the scaling window are *delocalized in ℓ^1 sense*, localized in ℓ^2 sense.

Theorem 2 [A non-resonant delocalized state for $\lambda < \sqrt{2}$]
For $\lambda < \sqrt{2}$, there is a sequence of energies satisfying $\lim_{M\to\infty} \mathcal{E}_M = -1$ such that within the scaling windows centered at $\mathcal{E}_M$:

1. There exists one eigenvalue for which the corresponding eigenfunction ψ_E is ℓ^2-delocalized [...]
2. All other eigenfunctions in the scaling window are ℓ^2-localized [...]

Theorem 3 [Localization]
Within any sequence of scaling windows centered asymptotically at

$$\mathcal{E} \;=\; \lim_{M\to\infty} \mathcal{E}_M \;\in (-\lambda,\lambda)\backslash\{-1,0\}\,,$$

1. the rescaled eigenvalues $(u_{M,n})$ coincide asymptotically, in probability, with the point process $(\omega_{M,n})$ of the rescaled potential values (the two being compared within scaling windows of fixed, but arbitrarily large size $[-W, W]$)

2. the eigenfunctions corresponding to energies with $|u_{M,n}| < W$ are all ℓ^2-localized in the sense that with asymptotically full probability all satisfy, for any $\gamma > 0$:

$$1 \;\leq\; \frac{\|\psi_{M,n}\|_2}{\|\psi_{M,n}\|_\infty} \;\leq\; 1 + \mathcal{O}\left(\frac{1}{M^{(\mathcal{E}/\lambda)^2(1-\gamma)}}\right) + \mathcal{O}\left(\frac{1}{M^{1/2-\gamma}}\right)\,.$$

Key elements of the proof

- Rank-one perturbation arguments yield the characteristic equation:

$$\text{Eigenvalues}: \quad \frac{1}{M}\sum_n \frac{1}{\kappa_M V_n - E} = 1 \quad (*)$$

$$\text{Eigenvectors}: \quad \psi_{j,E} = \frac{1}{\kappa_M V_j - E} \quad \text{up to normalization}$$

- To study the scaling limit we distinguish between the head contribution in (*), $S_{M,\omega}(u)$, and the tail sum, transforming (*) into:

$$\boxed{S_{M,\omega}(u) \;=\; M\Delta_M(\mathcal{E}) - T_{M,\omega}(u)} \;:=\; -R_{M,\omega}(u)$$

with

$$T_{M,\omega}(u) \;=\; \sum_n \frac{1[|\omega_n| \geq \ln M]}{\omega_{M,n} - u}$$

- Prove & apply some general results concerning limits of random Pick functions (aka Herglotz - Nevanlinna functions). In particular: the scaling limit of a function such as $R_{M,\omega}(u)$ is either:
  1. constant $\Rightarrow$ Šeba process & semi-delocalization,
  2. singular $(+\infty)$ or $(-\infty)$ $\Rightarrow$ localization, or
  3. singular with transition $\Rightarrow$ localization $+$ single deloc. state $(\mathcal{E} = -1, \lambda < \sqrt{2})$

Random Pick functions, and some facts about their limits

Pick class functions(*): functions $F : \mathbb{C}_+ \mapsto \mathbb{C}_+$ which are:
i) analytic in $\mathbb{C}_+$, and ii) satisfy $\operatorname{Im} F(x+iy) \geq 0$ for $y > 0$.

Such functions have the Herglotz representation:

$$F(z) \;=\; a_F z + b_F + \int \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) \mu_F(dx)$$

$P(a,b)$ - the subclass of Pick functions which are analytic in $(a,b) \subset \mathbb{R}$.

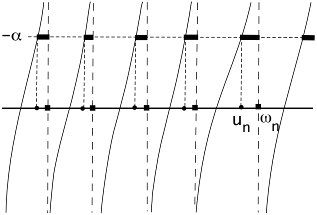
Pick, Löwner, Herglotz, Nevanlinna

Random Pick functions:
$\mu_F(dx)$ a random measure, e.g. point process, (a_F, b_F) may also be random.

The charact. eq. $\boxed{S_{M,\omega}(u) = -R_{M,\omega}(u)}$ relates two rather different examples:

1. $S_{M,\omega}(u)$: its spectral measure μ_S converges to a Poisson process
2. $R_{M,\omega}(u)$: is in $P(-L_M, L_M)$ for $L_M = \ln M \to \infty$

The “oscillatory part”



Prop 1: For any Pick function $S_\omega(x)$ which is stationary and ergodic under shifts, and of purely singular spectral measure, the value of $S_\omega(x)$ has the general Cauchy distribution ($\stackrel{\mathcal{D}}{=} aY + b$; Y Cauchy RV)

(See A.-Warzel '13, may have been know to Methuselah.)

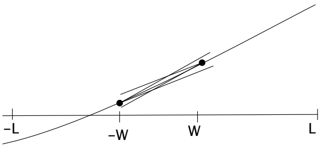
Among the interesting examples:

1. (periodic) the function $S_\theta(u) \;=\; \cot(u+\theta)$
2. (random, no level repulsion) the Poisson-Stieltjes function $S_\omega(u)$
3. (random, with level repulsion) the Wigner matrix resolvent $S(u) = \langle 0|\frac{\Delta_N(\mathcal{E})}{H_{\omega,N}-(\mathcal{E}+u\Delta_N(\mathcal{E}))}|0\rangle$

Linearity away from the spectrum

Lemma: Let $F(z)$ be a function in $P(-L, L)$. Then $\forall\ W < L/3$ and $u, u_0, u_1 \in [-W, W]$,:

$$\left| \frac{F(u) - F(u_0)}{u - u_0} - \frac{F(u_1) - F(u_0)}{u_1 - u_0} \right| \leq \ 2\frac{W}{L}\ \frac{F(u_1) - F(u_0)}{u_1 - u_0}$$



Prop. 2:(A-S-W) Functions $F_M \in P(-L_M, L_M)$ with $L_M \to \infty$ can only have one of the following 3 limits

i. $F(z) = az + b$,

ii. singular: $(+\infty)$ or $(+\infty)$,

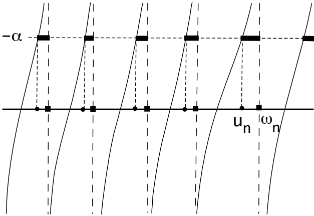
iii. singular with transition

and for (i) & (ii) convergence at two points suffices

The Šeba process

Let ω be the Poisson process of constant intensity 1.
The corresponding Stieltjes-Poisson random function

$$S_\omega(u) := \lim_{w\to\infty} \sum_n \frac{1[|\omega_n| \le w]}{\omega_n - u} \quad \text{(lim exists a.s.)}$$



For specified $\alpha \in [-\infty, \infty]$, denote by $\{u_{n,\omega}(\alpha)\}$ the solutions of:

$$\boxed{S_\omega(u) = \alpha}$$

Definition

We refer to the intertwined point process $(\{u_n, \omega_n\})$ as the Šeba point processes at level α.

Remarks:

- Limiting cases $\alpha = \pm\infty$: Poisson process
- Intermediate statistics with some level repulsion

Šeba 1990, Albeverio-Šeba 1991
Bogomolny/Gerland/Schmit 2001, Keating-Marklof-Winn 2003

Putting it all together

1. Proofs of Theorems 1 - 3 (the spectral characteristics of $H_{M,\omega}$)
2. validity of the heuristic criterion for resonant delocalization
3. different localization criteria
4. comments on operators with additional mixing terms
   (crossover to random matrix asymptotics)

Thanks for your attention