A classification of gapped Hamiltonians in $d = 1$

Sven Bachmann

Mathematisches Institut
Ludwig-Maximilians-Universität München

Joint work with Yoshiko Ogata

Spectral Days 2014

Quantum spin systems

- A lattice Γ of finite dimensional quantum systems (spins), with Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x\,, \qquad \Lambda \subset \Gamma\,, \text{finite}$$

- Observables on $\Lambda \subset \Gamma$: $\mathcal{A}_\Lambda = \mathcal{L}(\mathcal{H}_\Lambda)$
- Local Hamiltonian: a sum of short range interactions $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$$

- The Heisenberg dynamics:

$$\tau_\Lambda^t(A) = \exp(\mathrm{i}tH_\Lambda) A \exp(-\mathrm{i}tH_\Lambda)$$

States

The quasi-local algebra $\mathcal{A}_\Gamma$:

$$\mathcal{A}_\Gamma = \overline{\bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda}^{\|\cdot\|}$$

State ω: a positive, normalized, linear form on $\mathcal{A}_\Gamma$

- Finite volume Λ: $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$ and

$$\omega(A) = \mathrm{Tr}(\rho_\Lambda^\omega A)$$

  where ρ_Λ^ω is a density matrix
- Infinite systems Γ: No density matrix in general

But: Nets of states ω_Λ on $\mathcal{A}_\Lambda$ have weak-* accumulation points ω_Γ as $\Lambda \to \Gamma$: states in the thermodynamic limit

What is a quantum phase transition?

A simple answer: A phase transition at zero temperature

A slightly more precise answer: Consider:

- A smooth family of interactions $\Phi(s), s \in [0,1]$
- The associated family of Hamiltonians

$$H_\Lambda(s) = \sum_{X \in \Lambda} \Phi(X, s)$$

- Spectral gap above the ground state energy $\gamma_\Lambda(s)$ such that

$$\gamma_\Lambda(s) \geq \gamma(s) \begin{cases} > 0 & (s \neq s_c) \\ \sim C\,|s - s_c|^\mu & (s \to s_c) \quad \text{QPT} \end{cases}$$

- Associated singularity of the ground state projection $P_\Lambda(s)$

Basic question: What is a ground state phase?

Stability

Note: $\| \sum_{X\in\Lambda} \Phi(X) \| \gtrsim |\Lambda|$. But the dynamics exists

$$\lim_{\Lambda\to\Gamma} \mathrm{i}[H_\Lambda, A]$$

Pertubation:

$$H_\Lambda(s) = \sum_{X\in\Lambda} \big(\Phi(X) + s\Psi(X)\big)$$

If $\Psi(X)$ is local, *i.e.* $\Psi(X) = 0$ whenever $X \cap \Lambda_0^c \neq \emptyset$, then usually

- ▷ Dynamics $\tau^t_{\Gamma,s}$ as a perturbation of $\tau^t_{\Gamma,0}$
- ▷ Continuity of the spectral gap at $s = 0$
- ▷ Local perturbation of ground states
- ▷ Equilibrium states: $\|\omega_{\beta,s} - \omega_{\beta,0}\| \leq \kappa s$ as $s \to 0$

For translation invariant perturbations: No general stability results

Automorphic equivalence

Definition. *Two gapped H, H' are in the same phase if*

- ▷ *there is $s \mapsto \Phi(s)$, C^0 and piecewise C^1, with $\Phi(0) = \Phi, \Phi(1) = \Phi'$*
- ▷ *the Hamiltonians $H(s)$ are uniformly gapped*

$$\inf_{\Lambda \subset \Gamma, s \in [0,1]} \gamma_\Lambda(s) \geq \gamma > 0$$

The set of ground states on Γ: $\mathcal{S}_\Gamma(s)$.

Then there exists a continuous family of automorphism $\alpha_\Gamma^{s_1,s_2}$ of $\mathcal{A}_\Gamma$

$$\mathcal{S}_\Gamma(s_2) = \mathcal{S}_\Gamma(s_1) \circ \alpha_\Gamma^{s_1,s_2}$$

$\alpha_\Gamma^{s_1,s_2}$ is local: satisfies a Lieb-Robinson bound

Now: Invariants of the equivalence classes?

Frustration-free Hamiltonians in $d = 1$

Now $\Gamma = \mathbb{Z}$, and $\mathcal{H}_x \simeq \mathcal{H} = \mathbb{C}^n$
Consider spaces $\{\mathcal{G}_N\}_{N\in\mathbb{N}}$ such that $\mathcal{G}_N \subset \mathcal{H}^{\otimes N}$ and

$$\mathcal{G}_N = \bigcap_{x=0}^{N-m} \mathcal{H}^{\otimes x} \otimes \mathcal{G}_m \otimes \mathcal{H}^{\otimes(N-m-x)}$$

for some $m \in \mathbb{N}$; intersection property

Natural positive translation invariant interaction: G_m projection onto $\mathcal{G}_m$

$$\Phi(X) = \begin{cases} \tau_x(1 - G_m) & X = [x, x+m-1] \\ 0 & \text{otherwise} \end{cases}$$

By the intersection property: $\mathrm{Ker} H_{[1,N]} = \mathcal{G}_N$, parent Hamiltonian

Matrix product states

Consider $\mathbb{B} = (B_1, \dots, B_n)$, $B_i \in \mathcal{M}_k$ and two projections $p, q \in \mathcal{M}_k$

- A CP map $\mathcal{M}_k \to \mathcal{M}_k$:

$$\widehat{E}^{\mathbb{B}}(a) = \sum_{\mu=1}^{n} B_\mu a B_\mu^*$$

- $\mathbb{B} \in B_{n,k}(p,q)$ if
  1. Spectral radius of $\widehat{E}^{\mathbb{B}}$ is 1 and a non-degenerate eigenvalue
  2. No peripheral spectrum: other eigenvalues have $|\lambda| < 1$
  3. $e^{\mathbb{B}}$ and $\rho^{\mathbb{B}}$: right and left eigenvectors of $\widehat{E}^{\mathbb{B}}$: $pe^{\mathbb{B}}p$ and $q\rho^{\mathbb{B}}q$ invertible
- A map $\Gamma^{k,\mathbb{B}}_{N,p,q} : p\mathcal{M}_k q \to \mathcal{H}^{\otimes N}$:

$$\Gamma^{k,\mathbb{B}}_{N,p,q}(a) = \sum_{\mu_1, \dots \mu_N = 1}^{n} \mathrm{Tr}(paqB^*_{\mu_N} \cdots B^*_{\mu_1}) \psi_{\mu_1} \otimes \cdots \otimes \psi_{\mu_N}$$

Gapped parent Hamiltonian

Notation:

$$\mathcal{G}^{k,\mathbb{B}}_{N,p,q} = \mathrm{Ran}\left(\Gamma^{k,\mathbb{B}}_{N,p,q}\right) \subset \mathcal{H}^{\otimes N}$$

and parent Hamiltonian $H^{k,\mathbb{B}}_{N,p,q}$.

Proposition. *Assume that $\mathcal{G}^{k,\mathbb{B}}_{N,p,q}$ satisfies the intersection property. Then*

i. $H^{k,\mathbb{B}}_{N,p,q}$ is gapped

ii. $\mathcal{S}_{\mathbb{Z}}(H^{k,\mathbb{B}}_{\cdot,p,q}) = \{\omega^{\mathbb{B}}_{\infty}\}$

iii. Let $d_L = \dim(p), d_R = \dim(q)$. There are affine bijections:

$$\mathcal{E}\left(\mathcal{M}_{d_L}\right) \to \mathcal{S}_{(-\infty,-1]}(H^{k,\mathbb{B}}_{\cdot,p,q}), \qquad \mathcal{E}\left(\mathcal{M}_{d_R}\right) \to \mathcal{S}_{[0,\infty)}(H^{k,\mathbb{B}}_{\cdot,p,q})$$

i.e. Unique ground state on $\mathbb{Z}$, edge states determined by p, q

Bulk state

Given $\mathbb{B}$, for $A \in \mathcal{A}_{\{x\}}$,

$$\mathbb{E}_A^{\mathbb{B}}(b) := \sum_{\mu,\nu=1}^{n} \langle \psi_\mu, A\psi_\nu \rangle B_\mu^* b B_\nu$$

Note: $\widehat{\mathbb{E}}^{\mathbb{B}} = \mathbb{E}_1^{\mathbb{B}}(b)$.

$$\omega_\infty^{\mathbb{B}}(A_x \otimes \cdots \otimes A_y) = \rho^{\mathbb{B}} \left(\mathbb{E}_{A_x}^{\mathbb{B}} \circ \cdots \circ \mathbb{E}_{A_y}^{\mathbb{B}}(e^{\mathbb{B}}) \right)$$

- $\omega_\infty^{\mathbb{B}}(\Phi_{m,p,q}^{k,\mathbb{B}}(X)) = 0$: Ground state
- Exponential decay of correlations if $\sigma(\widehat{\mathbb{E}}^{\mathbb{B}}) \setminus \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$

$$\omega_\infty^{\mathbb{B}}(A_x \otimes 1^{\otimes |y-x-1|} \otimes A_y) = \rho^{\mathbb{B}} \left(\mathbb{E}_A^{\mathbb{B}} \circ (\widehat{\mathbb{E}}^{\mathbb{B}})^{|y-x-1|} \circ \mathbb{E}_B^{\mathbb{B}}(e) \right)$$

Edge states

Note: $\omega_\infty^{\mathbb{B}}(A_x \otimes \cdots \otimes A_y)$ extends to $\mathbb{Z}$:

$$\rho^{\mathbb{B}} \left(\mathbb{E}_{A_x}^{\mathbb{B}} \circ \cdots \circ \mathbb{E}_{A_y}^{\mathbb{B}}(e^{\mathbb{B}}) \right) = \rho^{\mathbb{B}} \left(\mathbb{E}_1^{\mathbb{B}} \circ \mathbb{E}_{A_x}^{\mathbb{B}} \circ \cdots \circ \mathbb{E}_{A_y}^{\mathbb{B}}(\mathbb{E}_1^{\mathbb{B}}(e^{\mathbb{B}})) \right)$$

For the same $\mathbb{E}_\cdot^{\mathbb{B}}$,

$$\omega_\varphi^{\mathbb{B}}(A_0 \otimes \cdots \otimes A_x) := \varphi \left((pe^{\mathbb{B}}p)^{-1/2} p \left(\mathbb{E}_{A_0}^{\mathbb{B}} \circ \cdots \circ \mathbb{E}_{A_x}^{\mathbb{B}}(e^{\mathbb{B}}) \right) p(pe^{\mathbb{B}}p)^{-1/2} \right)$$

for any state φ on $p\mathcal{M}_k p$, and $\omega_\varphi^{\mathbb{B}}(\Phi_{m,p,q}^{k,\mathbb{B}}(X)) = 0$
These extend to the right, but not to the left:

$$\mathcal{S}_{[0,\infty)}(H_{\cdot,p,q}^{k,\mathbb{B}}) \quad \longleftrightarrow \quad \mathcal{E}(\mathcal{M}_{d_L})$$

A complete classification

Theorem. *Let* $H := H^{k,\mathbb{B}}_{\cdot,p,q}$ *and* $H' := H^{k',\mathbb{B}'}_{\cdot,p',q'}$ *as in the proposition, with associated* (d_L, d_R)*, resp.* (d'_L, d'_R)*. Then,*

$$H \simeq H' \qquad \Longleftrightarrow \qquad (d_L, d_R) = (d'_L, d'_R)$$

Remark: No symmetry requirement

Proof by explicit construction of a gapped path of interactions $\Phi(s)$:

- on the fixed chain with $\mathcal{A}_{\{x\}} = \mathcal{B}(\mathbb{C}^n)$
- constant finite range
- translation invariant (no blocking)

Bulk product states

Very simple representatives of each phase:

Proposition. *Let* $n \geq 3$*, and* $(d_L, d_R) \in \mathbb{N}^2$*. Let* $k := d_L d_R$*. There exists* $\mathbb{B}$ *and projections* p, q *in* $\mathrm{Mat}_k(\mathbb{C})$ *such that*

- $\dim p = d_L$, $\dim q = d_R$
- $\mathbb{B} \in B_{n,k}(p, q)$
- $\mathcal{G}^{k,\mathbb{B}}_{m,p,q}$ *satisfy the intersection property*
- *the unique ground state* $\omega^{\mathbb{B}}_{\infty}$ *of the Hamiltonian* $H^{k,\mathbb{B}}_{\cdot,p,q}$ *on* $\mathbb{Z}$ *is the pure product state*

$$\omega^{\mathbb{B}}_{\infty}(A_x \otimes \cdots \otimes A_y) = \prod_{i=x}^{y} \langle \psi_1, A_i \psi_1 \rangle$$

Example: the AKLT model

- $\mathrm{SU}(2)$-invariant, antiferromagnetic spin-1 chain
- Nearest-neighbor interaction

$$H_{[a,b]}^{AKLT} = \sum_{x=a}^{b-1} \left[\frac{1}{2}\left(S_x \cdot S_{x+1}\right) + \frac{1}{6}\left(S_x \cdot S_{x+1}\right)^2 + \frac{1}{3} \right] = \sum_{x=a}^{b-1} P_{x,x+1}^{(2)}$$

where $P_{x,x+1}^{(2)}$ is the projection on the spin-2 space of $\mathcal{D}_1 \otimes \mathcal{D}_1$

- Uniform spectral gap γ of $H_{[a,b]}$, $\gamma > 0.137194$
- $H^{AKLT} = H^{2,\mathbb{B}}_{\cdot,1,1}$ with $\mathbb{B} \in B_{3,2}(1,1)$

$$B_1 = \begin{pmatrix} -\sqrt{1/3} & 0 \\ 0 & \sqrt{1/3} \end{pmatrix}, B_2 = \begin{pmatrix} 0 & -\sqrt{2/3} \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ \sqrt{2/3} & 0 \end{pmatrix}$$

- the AKLT model belongs to the phase $(2,2)$

About the proof

Recall:

$$\mathbb{B} \in B_{n,k}(p,q) \quad \longrightarrow \quad \widehat{\mathbb{E}}^{\mathbb{B}},\, \Gamma^{k,\mathbb{B}}_{N,p,q} \quad \longrightarrow \quad \mathcal{G}^{k,\mathbb{B}}_{N,p,q} \quad \longrightarrow \quad H^{k,\mathbb{B}}_{\cdot,p,q}, \omega^{\mathbb{B}}_{\infty}$$

and by the proposition

$$\text{Gap}(\widehat{\mathbb{E}}^{\mathbb{B}}) \quad \longrightarrow \quad \text{Gap}(H^{k,\mathbb{B}}_{\cdot,p,q})$$

Given $\mathbb{B} \in B_{n,k}(p,q), \mathbb{B}' \in B_{n,k'}(p',q')$, construct a path of gapped 'parent' Hamiltonians $H^{k,\mathbb{B}(s)}_{\cdot,p(s),q(s)}$ by

- embedding $\mathcal{M}_{k'} \hookrightarrow \mathcal{M}_k$ and interpolating
- interpolating $p(s), q(s)$: dimensions
- interpolating $\mathbb{B}(s)$, keeping spectral properties of $\widehat{\mathbb{E}}^{\mathbb{B}}$

Need pathwise connectedness of a certain subspace of $(\mathcal{M}_k)^{\times n}$

Primitive maps

$$\widehat{\mathbb{E}}^{\mathbb{B}} = \sum_{\mu=1}^{n} B_\mu \cdot B_\mu^*$$

i.e. $\{B_\mu\}$ are the Kraus operators

The spectral gap condition: Perron-Frobenius

- Irreducible positive map $\implies$
  1. Spectral radius r is a non-degenerate eigenvalue
  2. Corresponding eigenvector $e > 0$
  3. Eigenvalues λ with $|\lambda| = r$ are $r\mathrm{e}^{2\pi\mathrm{i}\alpha/\beta}$, $\alpha \in \mathbb{Z}/\beta\mathbb{Z}$
- A primitive map is an irreducible CP map with $\beta = 1$

Lemma. $\widehat{\mathbb{E}}^{\mathbb{B}}$ *is primitive iff there exists* $m \in \mathbb{N}$ *such that*

$$\mathrm{span}\,\{B_{\mu_1} \cdots B_{\mu_m} : \mu_i \in \{1, \dots, n\}\} = \mathcal{M}_k$$

Primitive maps

How to construct paths of primitive maps? Consider

$$Y_{n,k} := \left\{ \mathbb{B} : B_1 = \sum_{\alpha=1}^{k} \lambda_\alpha \left| e_\alpha \right\rangle \left\langle e_\alpha \right|, \quad \text{and} \quad \langle B_2 e_\alpha, e_\beta \rangle \neq 0 \right\}$$

with the choice

$$(\lambda_1, \dots, \lambda_k) \in \Omega := \{\lambda_i \neq 0, \lambda_i \neq \lambda_j, \lambda_i/\lambda_j \neq \lambda_k/\lambda_l\}$$

Then,

$$\left| e_\alpha \right\rangle \left\langle e_\beta \right| \in \operatorname{span}\{B_{\mu_1} \cdots B_{\mu_m} : \mu_i \in \{1, 2\}\}$$

for $m \geq 2k(k-1) + 3$.

Problem reduced to the pathwise connectedness of $\Omega \subset \mathbb{C}^k$

Use transversality theorem

Consequences

What we obtain:

$$\mathbb{B}(s) \in B_{n,k}(1,1) \subset B_{n,k}(p(s),q(s))$$

i.e. a good $\widehat{\mathbb{E}}^{\mathbb{B}(s)}$

For those:

- $\Gamma^{k,\mathbb{B}}_{m,p(s),q(s)}$ is injective $\quad\Rightarrow\quad \dim \mathcal{G}^{k,\mathbb{B}}_{m,p(s),q(s)} = d_R d_L$
- $\mathcal{G}^{k,\mathbb{B}}_{m,p(s),q(s)}$ satisfy the intersection property

i.e. a good path $H^{k,\mathbb{B}}_{\cdot,p(s),q(s)}$

Remarks

- More work at $s = 0, 1$, where the given $\mathbb{B}, \mathbb{B}' \notin Y_{n,k}$:
- Why is it hard? Because $\dim \mathcal{H} = n$ is fixed
- Simpler problem for $n \geq k^2$, i.e. by allowing periodic interactions
- Interaction range:

$$m_{min} = \max\{m, m', k^2 + 1, (k')^2 + 1\}$$

- all in all: $(d_L, d_R) = (d'_L, d'_R)$ is sufficient
- $(d_L, d_R) = (d'_L, d'_R)$ necessary: $H \simeq H'$ implies

$$\mathcal{S}_{[0,\infty)} = \mathcal{S}'_{[0,\infty)} \circ \alpha_{[0,\infty)}, \qquad \mathcal{S}_{(-\infty,-1]} = \mathcal{S}'_{(-\infty,-1]} \circ \alpha_{(-\infty,-1]}$$

and α is bijective

Concrete representatives

$$S_0(\lambda, d) = \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda^{d-1} \end{pmatrix}, \qquad S_+(d) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

Let

$$\begin{aligned} B_1 &= S_0(\lambda_R, d_R) \otimes S_0(\lambda_L, d_L) \\ B_2 &= S_+(d_R) \otimes S_0(\lambda_L, d_L) \\ B_3 &= S_0(\lambda_R, d_R) \otimes S_+(d_L) \\ B_i &= 0 \qquad \text{if } i \geq 3. \end{aligned}$$

Properties: $B_2^{d_R} = 0$, $B_3^{d_L} = 0$, and

$$B_1^* B_2^* = \lambda_R B_2^* B_1^*, \qquad B_1^* B_3^* = \lambda_L B_3^* B_1^*, \qquad B_2^* B_3^* = \left(\frac{\lambda_L}{\lambda_R}\right) B_3^* B_2^*.$$

Concrete spectrum

Simple consequence:

$$\widehat{\mathbb{E}}^{\mathbb{B}} = \mathbb{D} + \mathbb{N}_R + \mathbb{N}_L$$

with $\mathbb{D} = B_1 \cdot B_1^*$ diagonal, $\mathbb{N}_R = B_2 \cdot B_2^*$, $\mathbb{N}_L = B_3 \cdot B_3^*$, nilpotent, and

$$\mathbb{D}\mathbb{N}_R = \lambda_R^{-2}\mathbb{N}_R\mathbb{D}, \qquad \mathbb{D}\mathbb{N}_L = \lambda_L^{-2}\mathbb{N}_L\mathbb{D}, \qquad \mathbb{N}_R\mathbb{N}_L = (\lambda_R/\lambda_L)^2\mathbb{N}_L\mathbb{N}_R.$$

Then,

$$\sigma(\widehat{\mathbb{E}}^{\mathbb{B}}) = \sigma(\mathbb{D})$$

Spectral gap if $\lambda_L, \lambda_R \neq 1$

Product vacuum in the bulk

Vectors? Recall

$$\Gamma^{k,\mathbb{B}}_{N,p,q}(a) = \sum_{\mu_1,\dots\mu_N=1}^{n} \mathrm{Tr}(paqB^*_{\mu_N}\cdots B^*_{\mu_1})\psi_{\mu_1}\otimes\cdots\otimes\psi_{\mu_N}$$

The product $B_{\mu_1}\cdots B_{\mu_N}$ can have at most d_R-1 B_2's, and d_L-1 B_1's, so

$$\mathcal{G}^{k,\mathbb{B}}_{N,p,q} = \mathrm{span}\left\{\Gamma^{k,\mathbb{B}}_{N,p,q}(pB_2^\alpha B_3^\beta q)\right\}_{\alpha=0,\dots,d_R-1,\beta=0,\dots,d_L-1}$$

for $\alpha=\beta=0$, product vacuum:

$$\Gamma^{k,\mathbb{B}}_{N,p,q}(1) = \mathrm{Tr}(p1q(B_1^*)^m)\psi_1\otimes\cdots\otimes\psi_1$$

Conclusions

- Construction of gapped Hamiltonians from frustration-free states
- Unique ground state on $\mathbb{Z}$
- 'Tunable' number of edge states
- Complete classification by the asymmetric number of edge states (no symmetry)
- Each phase has a representative with a pure product state on $\mathbb{Z}$, no bulk-edge correspondence