Bulk-edge duality for topological insulators

Gian Michele Graf
ETH Zurich

Journées Méthodes Spectrales
Spectral Days 2014
CIRM
June 9, 2014

Bulk-edge duality for topological insulators

Gian Michele Graf
ETH Zurich

Journées Méthodes Spectrales
Spectral Days 2014
CIRM
June 9, 2014

joint work with Marcello Porta
thanks to Yosi Avron

Topological insulators: first impressions

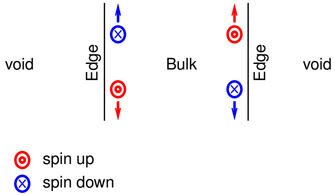
- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy

Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy
- Time-reversal invariant fermionic system

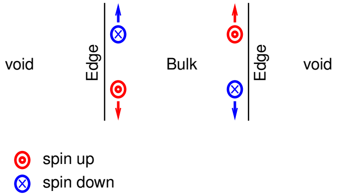
Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy
- Time-reversal invariant fermionic system



Topological insulators: first impressions

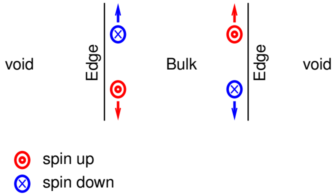
- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy
- Time-reversal invariant fermionic system



- Topology: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.

Topological insulators: first impressions

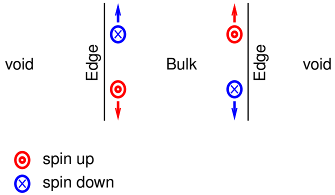
- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy
- Time-reversal invariant fermionic system



- Topology: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.
  Analogy: torus $\neq$ sphere (differ by genus).

Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy
- Time-reversal invariant fermionic system

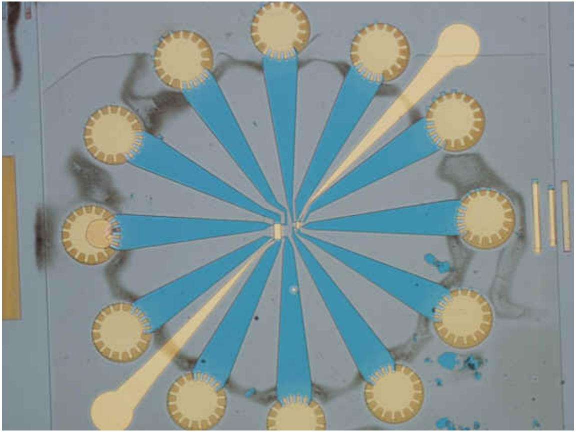


- Topology: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.
  Analogy: torus $\neq$ sphere (differ by genus).

Contributors to the field: Kane, Mele, Zhang, Moore; Fröhlich; Hasan

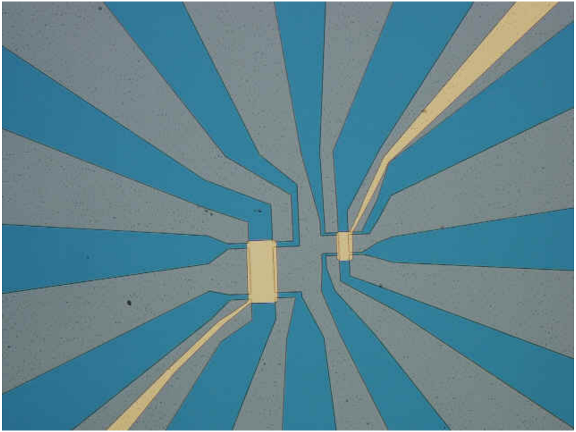
Pictures

Material: InAs/GaSb (quantum well); AlSb (barrier)



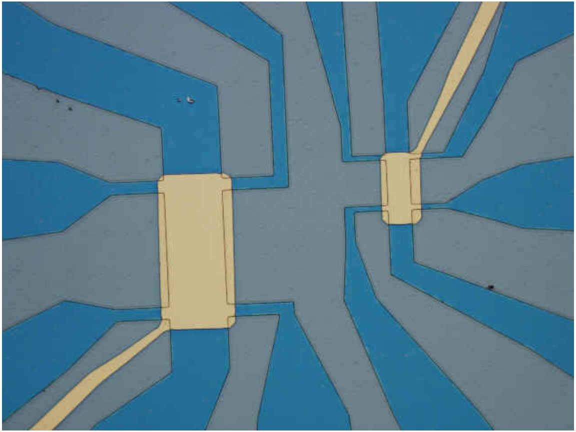
Courtesy: S. Müller, K. Ensslin

Pictures



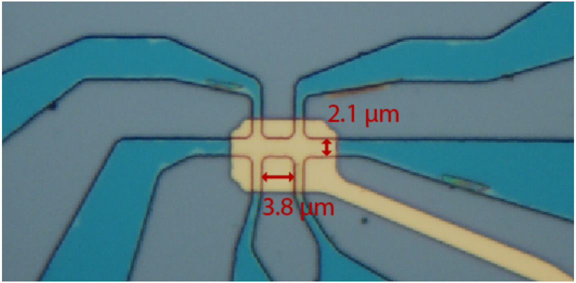
Courtesy: S. Müller, K. Ensslin

Pictures



Courtesy: S. Müller, K. Ensslin

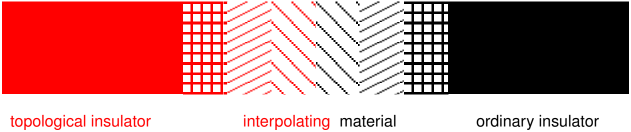
Pictures



Courtesy: S. Müller, K. Ensslin

Bulk-edge correspondence

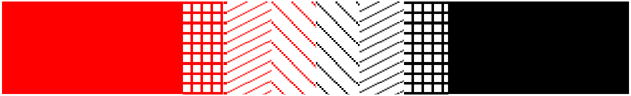
Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.

Bulk-edge correspondence

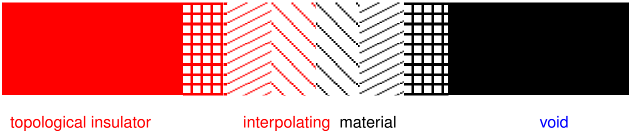
Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states

Bulk-edge correspondence

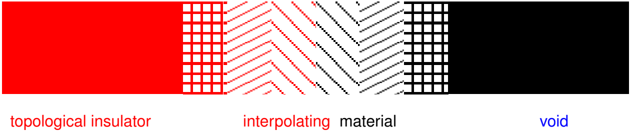
Deformation as interpolation in physical space:



- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states.

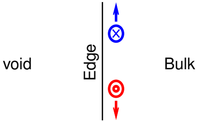
Bulk-edge correspondence

Deformation as interpolation in physical space:



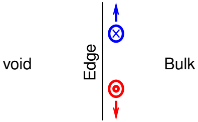
- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence



In a nutshell: Termination of bulk of a topological insulator implies edge states

Bulk-edge correspondence

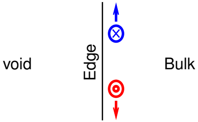


In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

Bulk-edge correspondence



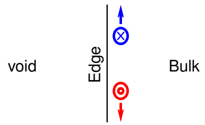
In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge.

Bulk-edge correspondence



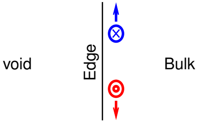
In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge.
- Bulk-edge duality: Can it be shown that the two indices agree?

Bulk-edge correspondence. Done?



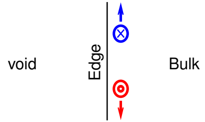
In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge. Yes, e.g. Kane and Mele.
- Bulk-edge duality: Can it be shown that the two indices agree? Schulz-Baldes et al.; Essin & Gurarie

Bulk-edge correspondence. Today



In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge. Done differently.
- Bulk-edge duality: Can it be shown that the two indices agree? Done differently.

Rueda de casino. Time 0′15″

Rueda de casino. Time 0′25″

Rueda de casino. Time $0'35''$

Rueda de casino. Time $0'44''$

Rueda de casino. Time 0′44.25″

Rueda de casino. Time $0'44.50''$

Rueda de casino. Time $0'44.75''$

Rueda de casino. Time $0'45''$

Rueda de casino. Time $0'45.25''$

Rueda de casino. Time $0'45.50''$

Rueda de casino. Time $0'46''$

Rueda de casino. Time $0'47''$

Rueda de casino. Time $0'55''$

Rueda de casino. Time 1′16″

Rueda de casino. Time 3′23″

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor
- are unlabeled points

Rules of the dance

Dancers

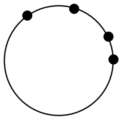
- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor
- are unlabeled points

There are dances which can not be deformed into one another.

What is the index that makes the difference?

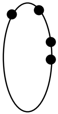
The index of a Rueda

A snapshot of the dance

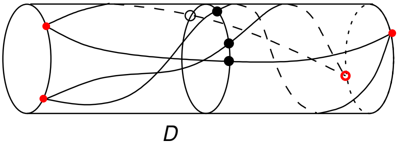


The index of a Rueda

A snapshot of the dance

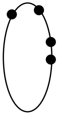


Dance D as a whole

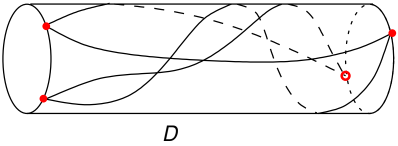


The index of a Rueda

A snapshot of the dance

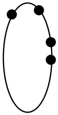


Dance D as a whole

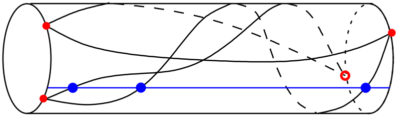


The index of a Rueda

A snapshot of the dance



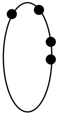
Dance D as a whole



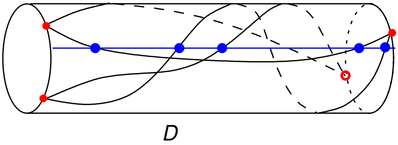
D

The index of a Rueda

A snapshot of the dance

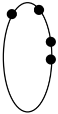


Dance D as a whole

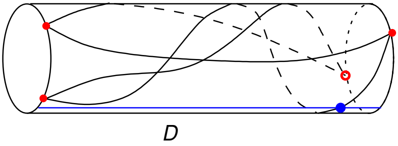


The index of a Rueda

A snapshot of the dance

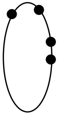


Dance D as a whole

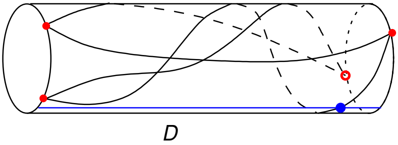


The index of a Rueda

A snapshot of the dance



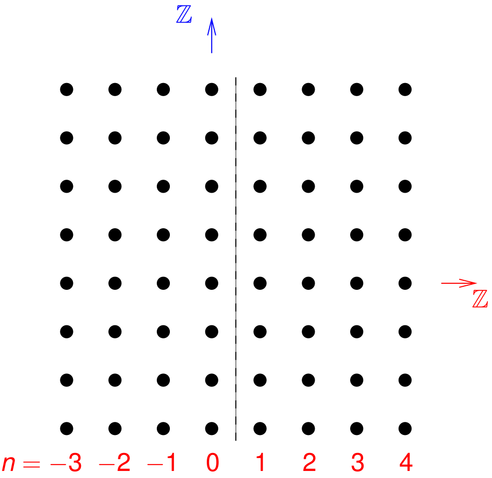
Dance D as a whole



$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

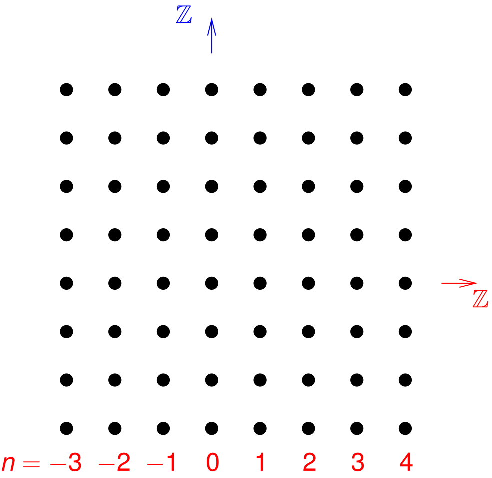
Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1:

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f.

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f., along with others (spin, ...)

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f., along with others (spin, ...), totalling N

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f., along with others (spin, ...), totalling N
- Bloch reduction by quasi-momentum $k \in S^1 := \mathbb{R}/2\pi\mathbb{Z}$

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f., along with others (spin, ...), totalling N
- Bloch reduction by quasi-momentum $k \in S^1 := \mathbb{R}/2\pi\mathbb{Z}$

End up with wave-functions $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^N)$ and Bulk Hamiltonian

$$\big(H(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

with
$V_n(k) = V_n(k)^* \in M_N(\mathbb{C})$ (potential)
$A(k) \in \mathrm{GL}(N)$ (hopping)

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)

- translation invariant in the vertical direction
- period may be assumed to be 1: sites within a period as labels of internal d.o.f., along with others (spin, ...), totalling N
- Bloch reduction by quasi-momentum $k \in S^1 := \mathbb{R}/2\pi\mathbb{Z}$

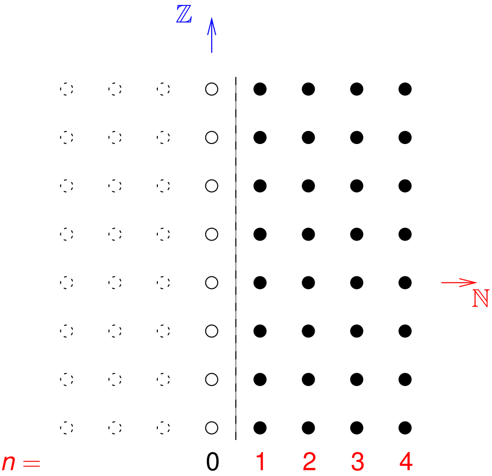
End up with wave-functions $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^N)$ and Bulk Hamiltonian

$$\big(H(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

with
$V_n(k) = V_n(k)^* \in M_N(\mathbb{C})$ (potential)
$A(k) \in \mathrm{GL}(N)$ (hopping) : Schrödinger eq. is the 2nd order difference equation

Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$

- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

$$\big(H^\sharp(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n^\sharp(k)\psi_n$$

Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$

- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

$$\big(H^\sharp(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n^\sharp(k)\psi_n$$

which

- agrees with Bulk Hamiltonian outside of collar near edge (width n_0)

$$V_n^\sharp(k) = V_n(k)\ , \qquad (n > n_0)$$

Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$

- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

$$\big(H^\sharp(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n^\sharp(k)\psi_n$$

which

- agrees with Bulk Hamiltonian outside of collar near edge (width n_0)
$$V_n^\sharp(k) = V_n(k)\,, \qquad (n > n_0)$$
- has Dirichlet boundary conditions: for $n = 1$ set $\psi_0 = 0$

Edge Hamiltonian

Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$

- translation invariant as before (hence Bloch reduction)

Wave-functions $\psi \in \ell^2(\mathbb{N}; \mathbb{C}^N)$ and Edge Hamiltonian

$$\left(H^\sharp(k)\psi\right)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n^\sharp(k)\psi_n$$

which

- agrees with Bulk Hamiltonian outside of collar near edge (width n_0)

$$V_n^\sharp(k) = V_n(k)\,, \qquad (n > n_0)$$

- has Dirichlet boundary conditions: for $n = 1$ set $\psi_0 = 0$

Note: $\sigma_{\text{ess}}(H^\sharp(k)) \subset \sigma_{\text{ess}}(H(k))$, but typically $\sigma_{\text{disc}}(H^\sharp(k)) \not\subset \sigma_{\text{disc}}(H(k))$

General assumptions

- Gap assumption: Fermi energy μ lies in a gap for all $k \in S^1$:

$$\mu \notin \sigma(H(k))$$

General assumptions

- Gap assumption: Fermi energy μ lies in a gap for all $k \in S^1$:
$$\mu \notin \sigma(H(k))$$
- Fermionic time-reversal symmetry: $\Theta : \mathbb{C}^N \to \mathbb{C}^N$
  - Θ is anti-unitary and $\Theta^2 = -1$;
  - Θ induces map on $\ell^2(\mathbb{Z}; \mathbb{C}^N)$, pointwise in $n \in \mathbb{Z}$;
  - For all $k \in S^1$,
$$H(-k) = \Theta H(k) \Theta^{-1}$$
    Likewise for $H^\sharp(k)$

Elementary consequences of $H(-k) = \Theta H(k) \Theta^{-1}$

- $\sigma(H(k)) = \sigma(H(-k))$. Same for $H^\sharp(k)$.

Elementary consequences of $H(-k) = \Theta H(k)\Theta^{-1}$

- $\sigma(H(k)) = \sigma(H(-k))$. Same for $H^\sharp(k)$.
- Time-reversal invariant points, $k = -k$, at $k = 0, \pi$.

Elementary consequences of $H(-k) = \Theta H(k) \Theta^{-1}$

- $\sigma(H(k)) = \sigma(H(-k))$. Same for $H^\sharp(k)$.
- Time-reversal invariant points, $k = -k$, at $k = 0, \pi$. There

$$H = \Theta H \Theta^{-1} \qquad (H = H(k) \text{ or } H^\sharp(k))$$

  Hence any eigenvalue is even degenerate (Kramers).

Elementary consequences of $H(-k) = \Theta H(k)\Theta^{-1}$

- $\sigma(H(k)) = \sigma(H(-k))$. Same for $H^\sharp(k)$.
- Time-reversal invariant points, $k = -k$, at $k = 0, \pi$. There

$$H = \Theta H \Theta^{-1} \qquad (H = H(k) \text{ or } H^\sharp(k))$$

Hence any eigenvalue is even degenerate (Kramers). Indeed

$$H\psi = E\psi \implies H(\Theta\psi) = E(\Theta\psi)$$

and $\Theta\psi = \lambda\psi$, ($\lambda \in \mathbb{C}$) is impossible:

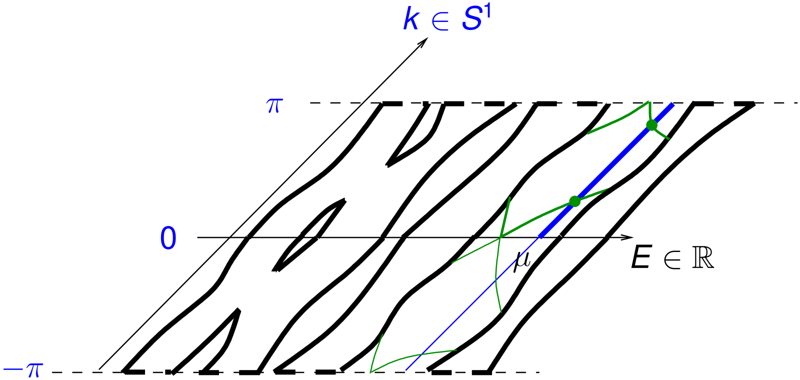
$$-\psi = \Theta^2\psi = \bar{\lambda}\Theta\psi = \bar{\lambda}\lambda\psi \qquad (\Rightarrow\Leftarrow)$$

Elementary consequences of $H(-k) = \Theta H(k) \Theta^{-1}$

- $\sigma(H(k)) = \sigma(H(-k))$. Same for $H^\sharp(k)$.
- Time-reversal invariant points, $k = -k$, at $k = 0, \pi$. There

$$H = \Theta H \Theta^{-1} \qquad (H = H(k) \text{ or } H^\sharp(k))$$

Hence any eigenvalue is even degenerate (Kramers).

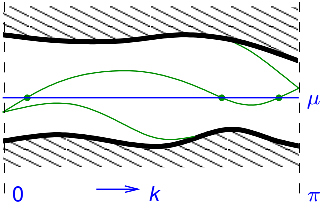


Bands, Fermi line (one half **fat**), edge states

The edge index

The spectrum of $H^\sharp(k)$

symmetric on $-\pi \le k \le 0$



Bands, Fermi line, edge states

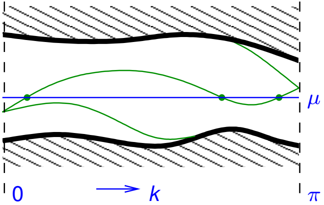
Definition: Edge Index

$$\mathcal{I}^\sharp = \text{parity of number of eigenvalue crossings}$$

The edge index

The spectrum of $H^\sharp(k)$

symmetric on $-\pi \le k \le 0$



Bands, Fermi line, edge states

Definition: Edge Index

$$\mathcal{I}^\sharp = \text{parity of number of eigenvalue crossings}$$

Collapse upper/lower band to a line and fold to a cylinder: Get rueda and its index.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n\in\mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n\in\mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

Let $z \notin \sigma(H(k))$. Then

$$E_{z,k} = \{\psi \mid \psi \text{ solution}, \psi_n \to 0, \ (n \to +\infty)\}$$

has

- $\dim E_{z,k} = N$.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n\in\mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

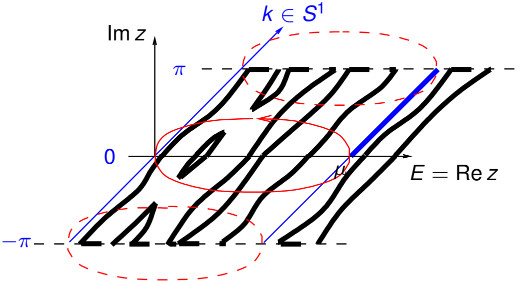
Let $z \notin \sigma(H(k))$. Then

$$E_{z,k} = \{\psi \mid \psi \text{ solution}, \psi_n \to 0, \ (n \to +\infty)\}$$

has

- $\dim E_{z,k} = N$.
- $E_{\bar{z},-k} = \Theta E_{z,k}$

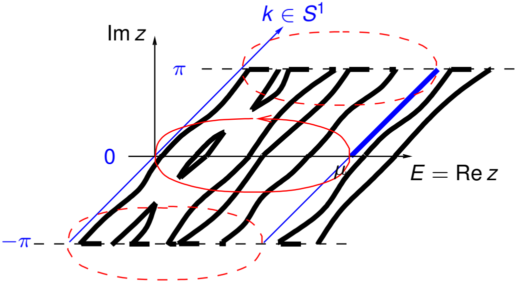
The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

The bulk index

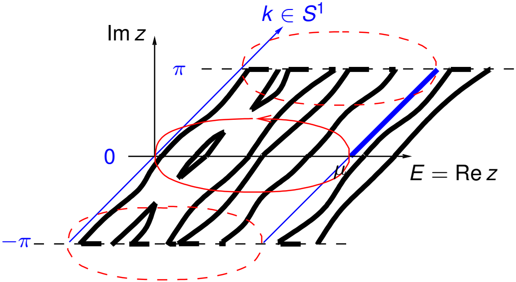


Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

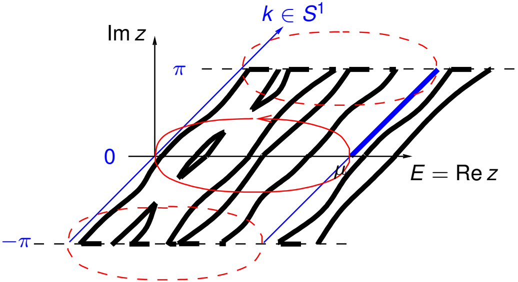
Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

Definition: Bulk Index

$$\mathcal{I} = \mathcal{I}(E)$$

The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

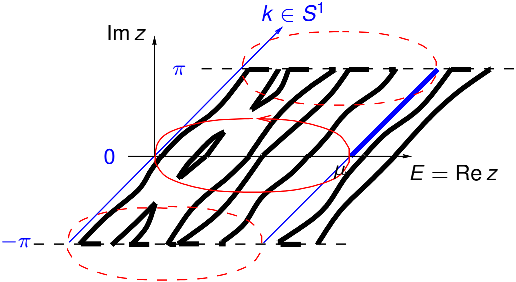
Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

Definition: Bulk Index

$$\mathcal{I} = \mathcal{I}(E)$$

What's behind the theorem? How is $\mathcal{I}(E)$ defined?

The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

Definition: Bulk Index

$$\mathcal{I} = \mathcal{I}(E)$$

What's behind the theorem? How is $\mathcal{I}(E)$ defined? Aside ... a rueda ...

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

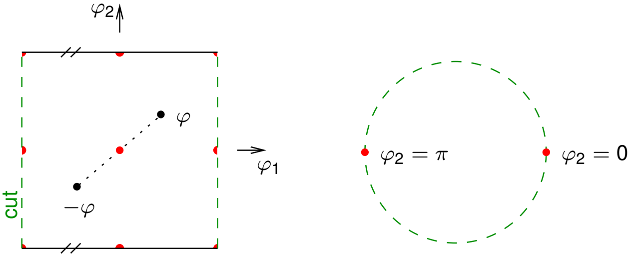
- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)

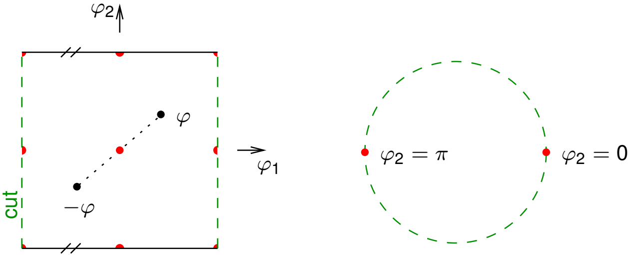


Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)



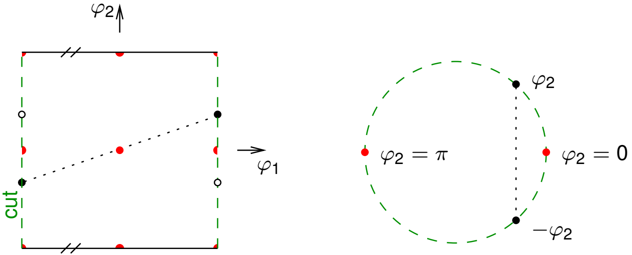
- a (compatible) section of the frame bundle of E

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)



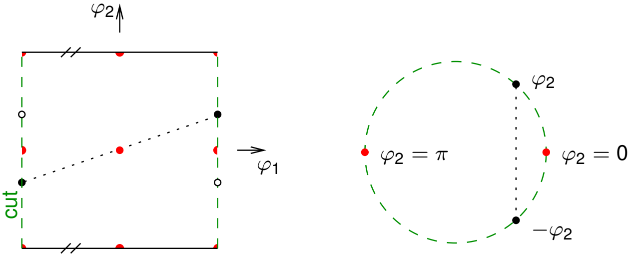
- a (compatible) section of the frame bundle of E
- the transition matrices $T(\varphi_2) \in \mathrm{GL}(N)$ across the cut

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

Sketch of proof: Consider

- torus $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ with cut (figure)



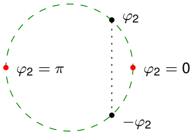
- a (compatible) section of the frame bundle of E
- the transition matrices $T(\varphi_2) \in \mathrm{GL}(N)$ across the cut

$$\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0\,, \qquad (\varphi_2 \in S^1)$$

with $\Theta_0 : \mathbb{C}^N \to \mathbb{C}^N$ antilinear, $\Theta_0^2 = -1$

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$

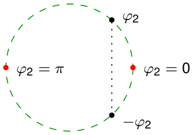


- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut $(0 \leq \varphi_2 \leq \pi)$ matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

$$\Theta_0 T = T^{-1}\Theta_0$$

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut $(0 \leq \varphi_2 \leq \pi)$ matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

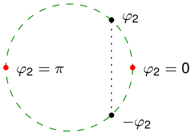
$$\Theta_0 T = T^{-1}\Theta_0$$

Eigenvalues of T come in pairs λ, $\bar{\lambda}^{-1}$:

$$\Theta_0(T - \lambda) = T^{-1}(1 - \bar{\lambda} T)\Theta_0$$

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut ($0 \leq \varphi_2 \leq \pi$) matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

$$\Theta_0 T = T^{-1}\Theta_0$$

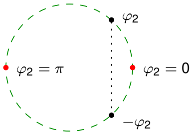
Eigenvalues of T come in pairs λ, $\bar{\lambda}^{-1}$:

$$\Theta_0(T - \lambda) = T^{-1}(1 - \bar{\lambda}T)\Theta_0$$

Phases $\lambda/|\lambda|$ pair up (Kramers degeneracy)

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- Only half the cut ($0 \leq \varphi_2 \leq \pi$) matters for $T(\varphi_2)$
- At time-reversal invariant points, $\varphi_2 = 0, \pi$,

$$\Theta_0 T = T^{-1}\Theta_0$$

Eigenvalues of T come in pairs λ, $\bar{\lambda}^{-1}$:

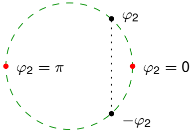
$$\Theta_0(T - \lambda) = T^{-1}(1 - \bar{\lambda}T)\Theta_0$$

Phases $\lambda/|\lambda|$ pair up (Kramers degeneracy)

- For $0 \leq \varphi_2 \leq \pi$, phases $\lambda/|\lambda|$ form a rueda, D

Time-reversal invariant bundles on the torus

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$



- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
- For $0 \leq \varphi_2 \leq \pi$, phases $\lambda/|\lambda|$ form a rueda, D

Definition (Index): $\mathcal{I}(E) := \mathcal{I}(D)$

Remark: $\mathcal{I}(E)$ agrees (in value) with the Pfaffian index of Kane and Mele.

. . . aside ends here.

Main result

Theorem Bulk and edge indices agree:

$$\mathcal{I} = \mathcal{I}^\sharp$$

Main result

Theorem Bulk and edge indices agree:

$$\mathcal{I} = \mathcal{I}^\sharp$$

$\mathcal{I} = +1$: ordinary insulator
$\mathcal{I} = -1$: topological insulator

Further results

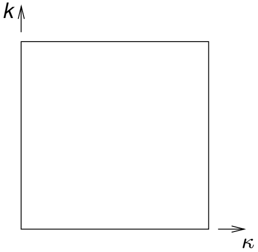
- Alternate formulation of bulk index
- Direct link to edge picture
- Application to graphene

Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k).

Alternate formulation of bulk index

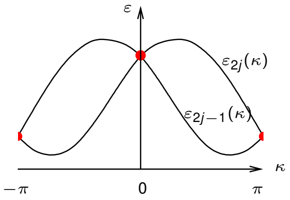
So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus



Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

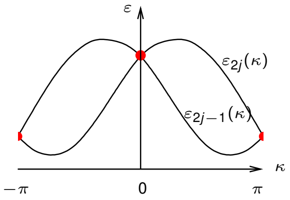
$k = 0, \pi$:



Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

$k = 0, \pi$:

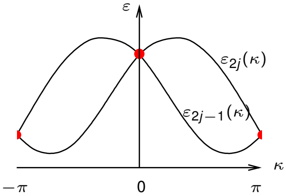


single bands can not be isolated; but pairs can.

Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

$k = 0, \pi$:

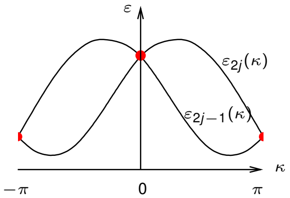


single bands can not be isolated; but pairs can.
If so: Bloch solutions for pair $(2j-1, 2j)$ form Bloch bundle E_j over Brillouin zone

Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

$k = 0, \pi$:



single bands can not be isolated; but pairs can. If so: Bloch solutions for pair $(2j-1, 2j)$ form Bloch bundle E_j over Brillouin zone

Theorem

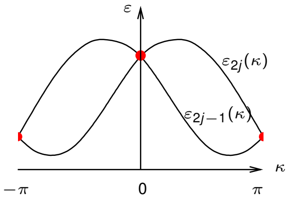
$$\mathcal{I} = \prod_j \mathcal{I}(E_j)$$

with product over filled pairs.

Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

$k = 0, \pi$:



single bands can not be isolated; but pairs can.
If so: Bloch solutions for pair $(2j-1, 2j)$ form Bloch bundle E_j over Brillouin zone

Theorem

$$\mathcal{I} = \prod_j \mathcal{I}(E_j)$$

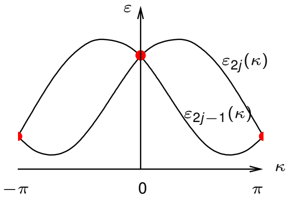
with product over filled pairs.

Note: Bulk solution are decaying to $n \to +\infty$, Bloch solutions are bounded

Alternate formulation of bulk index

So far, only periodicity along edge assumed (quasi-momentum k). Now: doubly periodic case (quasi-momenta k, κ): Brillouin zone serves as torus

$k = 0, \pi$:



single bands can not be isolated; but pairs can.
If so: Bloch solutions for pair $(2j-1, 2j)$ form Bloch bundle E_j over Brillouin zone

Theorem

$$\mathcal{I} = \prod_j \mathcal{I}(E_j)$$

with product over filled pairs.

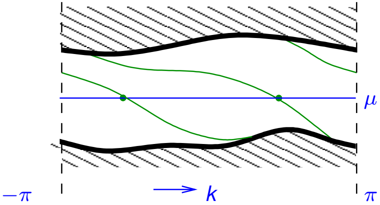
Note: Bulk solution are decaying to $n \to +\infty$, Bloch solutions are bounded
Proof using Bloch variety (Kohn)

A direct link to the edge

A direct link between indices of Bloch bundles and the edge index.

A direct link to the edge

A direct link between indices of Bloch bundles and the edge index. Simpler setting: Quantum Hall effect.



Definition: Edge Index

$$\mathcal{N}^\sharp = \text{signed number of eigenvalue crossings}$$

A direct link to the edge

A direct link between indices of Bloch bundles and the edge index. Simpler setting: Quantum Hall effect.

Definition: Edge Index

$$\mathcal{N}^{\sharp} = \text{signed number of eigenvalue crossings}$$

Bulk: $\mathrm{ch}(E_j)$ is the Chern number of the Bloch bundle E_j of the j-th band.

A direct link to the edge

A direct link between indices of Bloch bundles and the edge index. Simpler setting: Quantum Hall effect.

Definition: Edge Index

$$\mathcal{N}^\sharp = \text{signed number of eigenvalue crossings}$$

Bulk: $\mathrm{ch}(E_j)$ is the Chern number of the Bloch bundle E_j of the j-th band.

Duality:

$$\mathcal{N}^\sharp = \sum_j \mathrm{ch}(E_j)$$

with sum over filled bands.

A direct link to the edge

A direct link between indices of Bloch bundles and the edge index. Simpler setting: Quantum Hall effect.

Definition: Edge Index

$$\mathcal{N}^\sharp = \text{signed number of eigenvalue crossings}$$

Bulk: $\mathrm{ch}(E_j)$ is the Chern number of the Bloch bundle E_j of the j-th band.

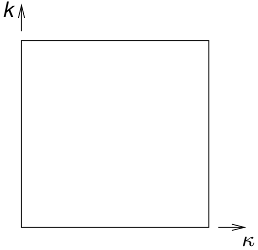
Duality:

$$\mathcal{N}^\sharp = \sum_j \mathrm{ch}(E_j)$$

with sum over filled bands.

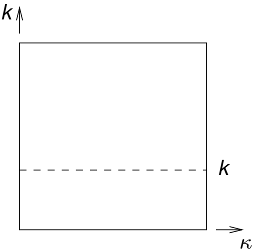
(cf. Hatsugai) Here via scattering and Levinson's theorem.

Duality via scattering

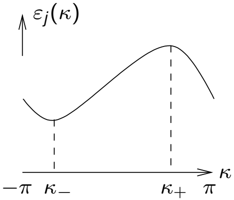


Brillouin zone $\ni (\kappa, k)$
Energy band $\varepsilon_j(\kappa, k)$

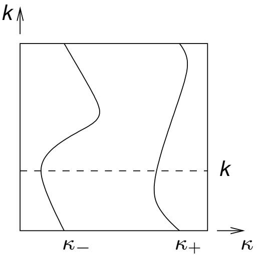
Duality via scattering



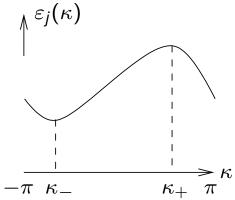
Minima $\kappa_-(k)$ and maxima $\kappa_+(k)$ of energy band $\varepsilon_j(\kappa, k)$ in κ at fixed k



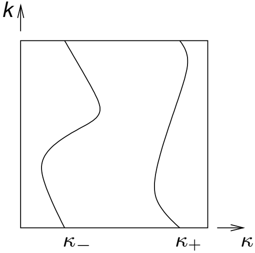
Duality via scattering



Minima $\kappa_-(k)$ and maxima $\kappa_+(k)$ of energy band $\varepsilon_j(\kappa, k)$ in κ at fixed k

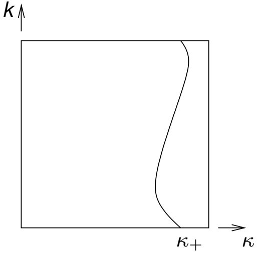


Duality via scattering



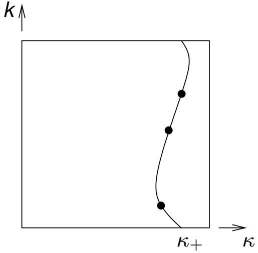
Minima $\kappa_-(k)$ and maxima $\kappa_+(k)$ of energy band $\varepsilon_j(\kappa, k)$ in κ at fixed k

Duality via scattering



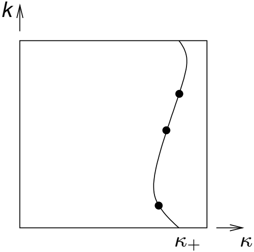
Maxima $\kappa_+(k)$

Duality via scattering

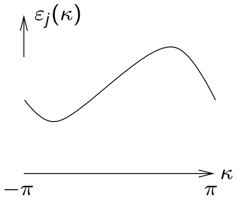


Maxima $\kappa_+(k)$ with semi-bound states (to be explained)

Duality via scattering

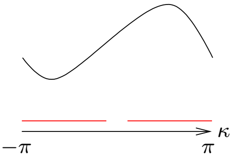


Duality via scattering



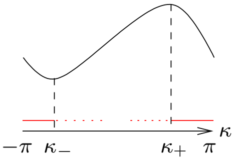
At fixed k: Energy band $\varepsilon_j(\kappa, k)$ and the line bundle E_j of Bloch states

Duality via scattering



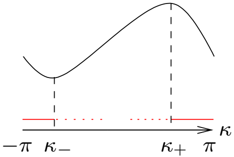
Line indicates choice of a section $|\kappa\rangle$ of Bloch states (from the given band). No global section in $\kappa \in \mathbb{R}/2\pi\mathbb{Z}$ is possible, as a rule.

Duality via scattering

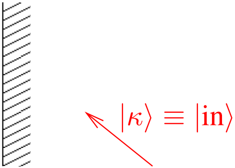


States $|\kappa\rangle$ above the solid line are left movers ($\varepsilon_j'(\kappa) < 0$)

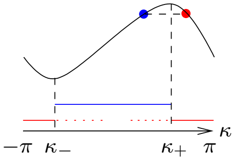
Duality via scattering



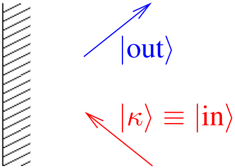
They are incoming asymptotic (bulk) states for scattering at edge (from inside)



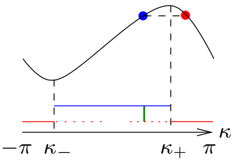
Duality via scattering

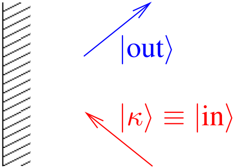


Scattering determines section $|\mathrm{out}\rangle$ of right movers above line



Duality via scattering



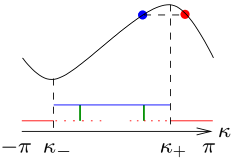


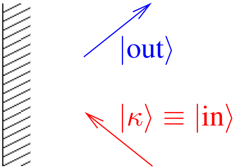
Scattering matrix

$$|\text{out}\rangle = S_+ |\kappa\rangle$$

as relative phase between two sections of the same fiber (near κ_+)

Duality via scattering





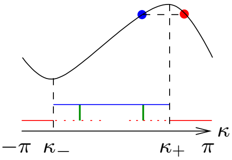
Scattering matrix

$$|\mathrm{out}\rangle = S_+ |\kappa\rangle$$

as relative phase between two sections of the same fiber (near κ_+)

Likewise S_- near κ_-.

Duality via scattering

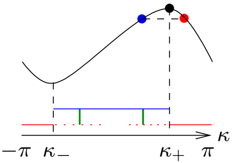


Chern number computed by sewing

$$\mathrm{ch}(E_j) = \mathcal{N}(S_+) - \mathcal{N}(S_-)$$

with $\mathcal{N}(S_\pm)$ the winding of
$S_\pm = S_\pm(k)$ as $k = 0 \ldots \pi$.

Duality via scattering



As $\kappa \to \kappa_+$, whence

$$|\text{in}\rangle = |\kappa\rangle \to |\kappa_+\rangle \qquad |\text{out}\rangle = S_+|\kappa\rangle \to |\kappa_+\rangle \text{ (up to phase)}$$

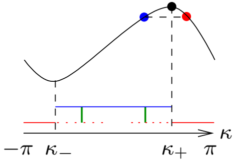
their limiting span is that of

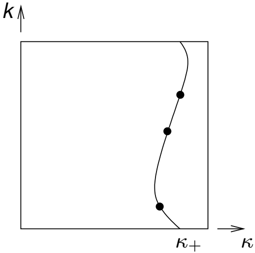
$$|\kappa_+\rangle, \quad \left.\frac{d|\kappa\rangle}{d\kappa}\right|_{\kappa_+}$$

(bounded, resp. unbounded in space). The span contains the limiting scattering state $|\psi\rangle \propto |\text{in}\rangle + |\text{out}\rangle$.

If (exceptionally) $|\psi\rangle \propto |\kappa_+\rangle$ then $|\psi\rangle$ is a semi-bound state.

Duality via scattering





As a function of *k*, semi-bound states occur exceptionally.

Levinson's theorem

Recall from two-body potential scattering: The scattering phase at threshold equals the number of bound states

$$\arg S\big|_{E=0+} = 2\pi N$$

Levinson's theorem

Recall from two-body potential scattering: The scattering phase at threshold equals the number of bound states

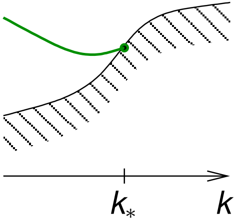
$$\arg S\big|_{E=0+} = 2\pi N$$

N changes with the potential V when bound state reaches threshold (semi-bound state $\equiv$ incipient bound state)

Levinson's theorem (relative version)

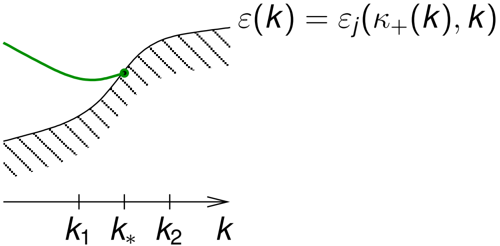
Spectrum of edge Hamiltonian

$$\varepsilon(k) = \varepsilon_j(\kappa_+(k), k)$$



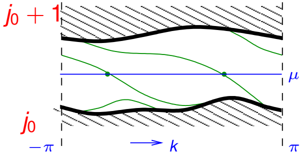
Levinson's theorem (relative version)

Spectrum of edge Hamiltonian



$$\lim_{\delta \to 0} \arg S_+(\varepsilon(k) - \delta)\Big|_{k_1}^{k_2} = \pm 2\pi$$

Proof



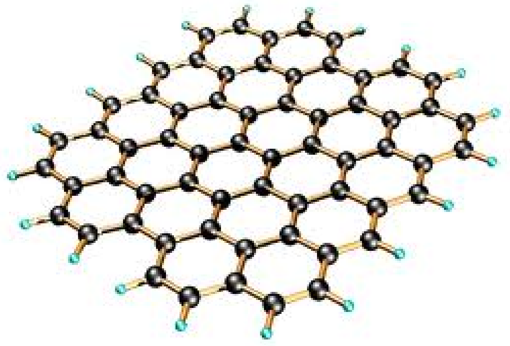
$$\mathcal{N}^\sharp = \mathcal{N}(S_+^{(j_0)}) \quad (= \mathcal{N}(S_-^{(j_0+1)}))$$

$$= \sum_{j=0}^{j_0} \mathcal{N}(S_+^{(j)}) - \mathcal{N}(S_-^{(j)})$$

$$= \sum_{j=0}^{j_0} \mathrm{ch}(E_j)$$

$(\mathcal{N}(S_-^{(1)}) = 0)$

An application: Quantum Hall in graphene

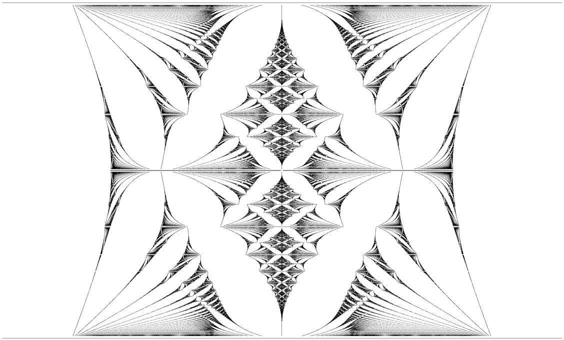


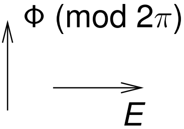
An application: Quantum Hall in graphene

Hamiltonian: Nearest neighbor hopping with flux Φ per plaquette.

An application: Quantum Hall in graphene

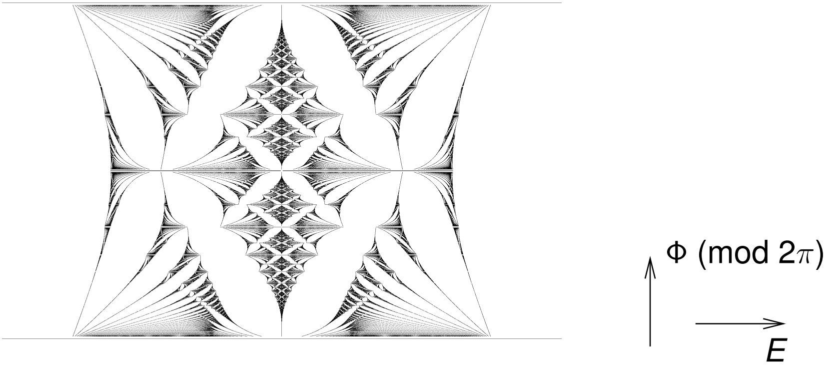
Spectrum in black





An application: Quantum Hall in graphene

Spectrum in black



What is the Hall conductance (Chern number) in any white point?

An application: Quantum Hall in graphene

What is the Hall conductance (Chern number) s in any white point?

Bulk approach (Thouless): If $\Phi = p/q$, (p, q coprime) then

$$r = sp + tq$$

where:

- r number of bands below Fermi energy
- s, t integers

s is so determined only modulo q.

An application: Quantum Hall in graphene

What is the Hall conductance (Chern number) s in any white point?

Bulk approach (Thouless): If $\Phi = p/q$, (p, q coprime) then

$$r = sp + tq$$

where:

- r number of bands below Fermi energy
- s, t integers

s is so determined only modulo q.

For square lattice, $s \in (-q/2, q/2)$.

An application: Quantum Hall in graphene

What is the Hall conductance (Chern number) s in any white point?

Bulk approach (Thouless): If $\Phi = p/q$, (p, q coprime) then

$$r = sp + tq$$

where:

- r number of bands below Fermi energy
- s, t integers

s is so determined only modulo q.

For square lattice, $s \in (-q/2, q/2)$. Not for other lattices.

An application: Quantum Hall in graphene

What is the Hall conductance (Chern number) s in any white point?

Bulk approach (Thouless): If $\Phi = p/q$, (p, q coprime) then

$$r = sp + tq$$

where:

- r number of bands below Fermi energy
- s, t integers

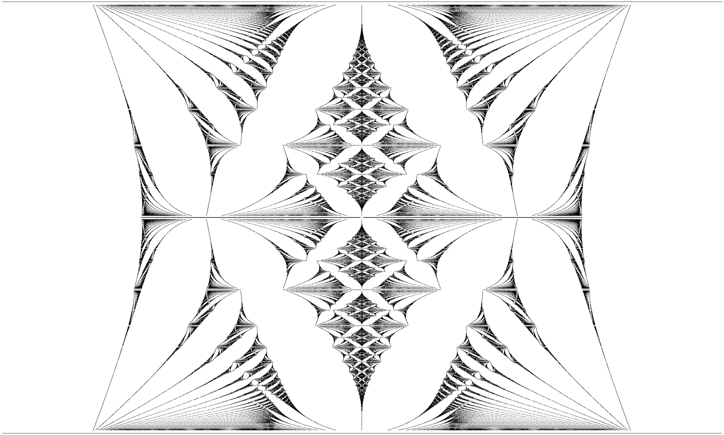
s is so determined only modulo q.

For square lattice, $s \in (-q/2, q/2)$. Not for other lattices.

$\to$ Edge approach (with Agazzi, Eckmann), method by Schulz-Baldes et al.

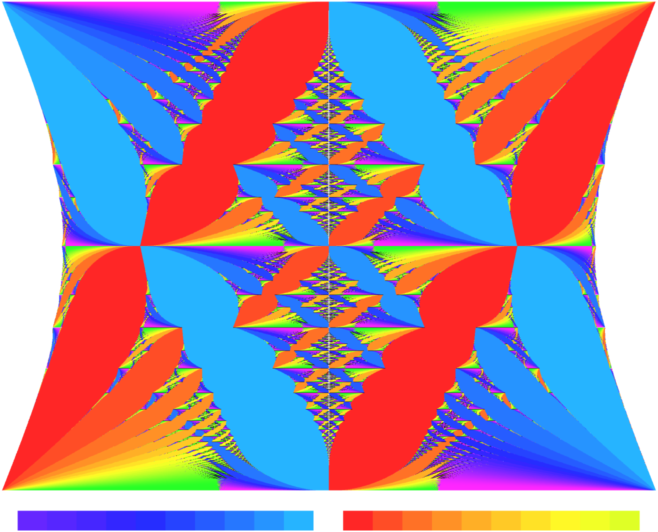
The colors of graphene

What is the Hall conductance (Chern number) in any white point?



The colors of graphene

What is the Hall conductance (Chern number) in any white point?



Summary

Bulk = Edge

$$\mathcal{I} = \mathcal{I}^\sharp$$

Summary

Bulk = Edge

$$\mathcal{I} = \mathcal{I}^\sharp$$

- The bulk and the indices of a topological insulator (of reduced symmetry) are indices of suitable ruedas
- In case of full translational symmetry, bulk index can be defined and linked to edge in other ways
- Application (Quantum Hall): graphene
- Three dimensions ...
- Open questions: No periodicity (disordered case)?