Asymptotic completeness for superradiant Klein-Gordon equations and applications to the De Sitter Kerr metric. arXiv:1405.5304

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1 The wave equation on the De Sitter Kerr metric

1.1 De Sitter Kerr metric in Boyer-Lindquist coordinates

$\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, with spacetime metric

$$
\begin{aligned}
g &= \frac{\Delta_r - a^2\sin^2\theta\Delta_\theta}{\lambda^2\rho^2}dt^2 + \frac{2a\sin^2\theta((r^2+a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta_r)}{\lambda^2\rho^2}dtd\varphi \\
&- \frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\sin^2\theta\sigma^2}{\lambda^2\rho^2}d\varphi^2, \\
\rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2Mr, \\
\Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2\cos^2\theta, \; \sigma^2 = (r^2+a^2)^2\Delta_\theta - a^2\Delta_r\sin^2\theta, \; \lambda = 1 + \frac{1}{3}\Lambda a^2.
\end{aligned}
$$

$\Lambda > 0$: cosmological constant, $M > 0$: masse, a : angular momentum per unit masse.

- $\rho^2 = 0$ is a curvature singularity, $\Delta_r = 0$ are coordinate singularities. $\Delta_r > 0$ on some open interval $r_- < r < r_+$. $r = r_-$: black hole horizon, $r = r_+$ cosmological horizon.
- ∂_φ and ∂_t are Killing. There exist $r_1(\theta)$, $r_2(\theta)$ s. t. ∂_t is
  - timelike on $\{(t, r, \theta, \varphi) : r_1(\theta) < r < r_2(\theta)\}$,
  - spacelike on $\{(t, r, \theta, \varphi) : r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi : r_2(\theta) < r < r_+\} =: \mathcal{E}_- \cup \mathcal{E}_+$.

  The regions $\mathcal{E}_-$, $\mathcal{E}_+$ are called ergospheres.

1.2 The wave equation on the De Sitter Kerr metric

We now consider the unitary transform

$$U: \begin{array}{ccc} L^2(\mathcal{M}; \frac{\sigma^2}{\Delta_r\Delta_\theta}drd\omega) & \to & L^2(\mathcal{M}; drd\omega) \\ \psi & \mapsto & \frac{\sigma}{\sqrt{\Delta_r\Delta_\theta}}\psi \end{array}$$

If ψ fulfills $(\Box_g + m^2)\psi = 0$, then $u = U\psi$ fulfills

$$(\partial_t^2 - 2ik\partial_t + h)u = 0. \tag{1}$$

with

$$\begin{aligned} k &= \frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)}{\sigma^2}D_\varphi, \\ h &= -\frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &\quad - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2. \end{aligned}$$

h is not positive inside the ergospheres. This entails that the natural conserved quantity

$$\tilde{\mathcal{E}}(u) = \|\partial_t u\|^2 + (hu|u)$$

is not positive.

1.3 3+1 decomposition, energies, Killing fields

Let $v = e^{-ikt}u$. Then u is solution of (1) if and only if v is solution of

$$(\partial_t^2 + h(t))v = 0, \quad h(t) = e^{-ikt} h_0 e^{ikt}, \quad h_0 = h + k^2 \geq 0.$$

Natural energy :

$$\|\partial_t v\|^2 + (h(t)v|v).$$

Rewriting for u:

$$\dot{\mathcal{E}}(u) = \|(\partial_t - ik)u\|^2 + (h_0 u|u).$$

This energy is positive, but may grow in time $\rightarrow$ superradiance.

Remark

$k = \Omega D_\varphi$ and Ω has finite limits $\Omega_{l/r}$ when $r \rightarrow r_\mp$. These limits are called angular velocities of the horizons. The Killing fields $\partial_t - \Omega_{l/r}\partial_\varphi$ on the De Sitter Kerr metric are timelike close to the black hole (l) resp. cosmological (r) horizon. Working with these Killing fields rather than with ∂_t leads to the conserved energies :

$$\tilde{\mathcal{E}}_{l/r}(u) = \|(\partial_t - \Omega_{l/r}\partial_\varphi)u\|^2 + (h_0 - (k - \Omega_{l/r}D_\varphi)^2 u|u).$$

Note that in the limit $k \rightarrow \Omega_{l/r}D_\varphi$ the expressions of $\dot{\mathcal{E}}(u)$ and $\tilde{\mathcal{E}}_{l/r}(u)$ coincide.

1.4 Asymptotic dynamics

Regge-Wheeler type coordinate $\frac{dx}{dr} = \frac{r^2+a^2}{\Delta_r}$.

$x \pm t = const.$ along principal null geodesics.

Unitary transform:

$$\mathcal{V} : \begin{array}{rcl} L^2(\mathbb{R}_{(r_-,r_+)} \times S^2) & \to & L^2(\mathbb{R} \times S^2, dxd\omega), \\ v(r,\omega) & \mapsto & \sqrt{\frac{\Delta_r}{r^2+a^2}}\, v(r(x),\omega). \end{array}$$

Asymptotic equations :

(2) $$(\partial_t^2 - 2\Omega_{l/r}\partial_\varphi\partial_t + h_{l/r})u_{l/r} = 0,$$
$$h_{l/r} = \Omega_{l/r}^2\partial_\varphi^2 - \partial_x^2.$$

The conserved quantities :

$$\begin{aligned} &\|(\partial_t - i\Omega_{l/r}D_\varphi)u_{l/r}\|^2 + ((h_{l/r} - \Omega_{l/r}^2\partial_\varphi^2)u_{l/r}|u_{l/r}) \\ = \quad &\|(\partial_t - i\Omega_{l/r}D_\varphi)u_{l/r}\|^2 + (-\partial_x^2 u_{l/r}|u_{l/r}) \end{aligned}$$

are positive.

Question : Can we compare the solutions of (1) to solutions of (2) for large times ?

1.5 Previous work

a) Scattering theory without positive conserved energy

Kako, C. Gérard, Georgescu-Gérard-H.

b) Superradiance

Bachelot (d=1).

c) Scattering theory on Kerr

H, H-Nicolas

d) Decay of the local energy on (De Sitter) Kerr spacetimes

Andersson-Blue, Bony-H ($a = 0$) , Dafermos-Rodnianski, Dyatlov, Finster-Kamran-Smoller-Yau,Tataru-Tohaneanu, Vasy, Zworski-Sa Barreto ($a = 0$),...

e) Hawking effect

Bachelot ($a = 0$),... , H (fermions, $a \neq 0$)

2.1 The abstract equation

$\mathcal{H}$ Hilbert space. h, k selfadjoint, $k \in \mathcal{B}(\mathcal{H})$.

(3) $$\left\{\begin{array}{rcl}(\partial_t^2 - 2ik\partial_t + h)u & = & 0,\\ u|_{t=0} & = & u_0,\\ \partial_t u|_{t=0} & = & u_1.\end{array}\right.$$

Hyperbolic equation

(A1) $$h_0 := h + k^2 \geq 0.$$

Formally $u = e^{izt}v$ solution if and only if

$$p(z)v = 0$$

with $p(z) = h_0 - (k - z)^2 = h + z(2k - z)$, $z \in \mathbb{C}$. $p(z)$ is called the quadratic pencil.

Conserved quantities

$$\langle u|u\rangle_\ell := \|u_1 - \ell u_0\|^2 + (p(\ell)u_0|u_0),$$

where $p(\ell) = h_0 - (k - \ell)^2$. Conserved by the evolution, but in general not positive definite, because none of the operators $p(\ell)$ is in general positive.

2.2 Spaces

$\mathcal{H}^i$: scale of Sobolev spaces associated to h_0.

(A2)

$$\left\{\begin{array}{c} 0 \notin \sigma_{pp}(h_0);\ k, h_0^{1/2} k h_0^{-1/2} \in \mathcal{B}(\mathcal{H}); \\ \forall z \in \mathbb{C} \setminus \mathbb{R},\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})} \lesssim |\mathrm{Im} z|^{-K_0},\ K_0 > 0. \\ \exists M_0 > 0,\ \forall |z| \geq M_0 \|k\|_{\mathcal{B}(\mathcal{H})},\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})} \lesssim \frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}}. \end{array}\right.$$

Homogeneous energy spaces

$$\dot{\mathcal{E}} = \Phi(k) h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \dot{\mathcal{E}}^* = \Phi(k)\mathcal{H} \oplus h_0^{1/2}\mathcal{H}, \quad \Phi(k) = \begin{pmatrix} \mathbb{1} & 0 \\ k & \mathbb{1} \end{pmatrix}.$$

where $\dot{\mathcal{E}}$ is equipped with the norm $\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}} = \|u_1 - k u_0\|^2 + (h_0 u_0 | u_0)$. We identify $\dot{\mathcal{E}}^*$ with the dual of $\dot{\mathcal{E}}$ with the help of the charge $\langle . | . \rangle = (u_0 | v_1 - k v_0) + (u_1 - k u_0 | v_0)$.

Lemma

For all $\ell \in \mathbb{R}$, $\langle . | . \rangle_\ell$ is continuous with respect to the norm $\|.\|_{\dot{\mathcal{E}}}$ if and only if $h_0 \gtrsim (k-\ell)^2$ in the sense of quadratic forms on $D(h_0)$.

Remark

In the case of the De Sitter Kerr metric this condition is not fulfilled.

2.3 Energy Klein Gordon operators

$$\begin{aligned}
\psi &= (u, \frac{1}{i}\partial_t u), \quad (\partial_t - iH)\psi = 0, \quad H = \begin{pmatrix} 0 & 1\!\!1 \\ h & 2k \end{pmatrix}, \\
(H-z)^{-1} &= p^{-1}(z) \begin{pmatrix} z-2k & 1\!\!1 \\ h & z \end{pmatrix}, \\
\rho(h,k) &:= \{z \in \mathbb{C} \mid p(z) : \langle h_0 \rangle^{-\frac{1}{2}} \mathcal{H} \tilde{\rightarrow} \langle h_0 \rangle^{\frac{1}{2}} \mathcal{H}\}.
\end{aligned}$$

Klein Gordon operator on the homogeneous energy space

$$D(\dot{H}) = \Phi(k)((h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}) \oplus \langle h_0 \rangle^{-1/2}\mathcal{H}),$$

$$\exists C_0 > 0,\ \rho(\dot{H}) \cap (\mathbb{C} \setminus (-C_0, C_0)) = \rho(h,k) \cap (\mathbb{C} \setminus (-C_0, C_0)),$$

$$\dot{R}(z) := (\dot{H} - z)^{-1}.$$

Gauge transformations

$$v = e^{-it\ell} u \quad (\partial_t - ik)^2 u + h_0 u = 0 \Leftrightarrow (\partial_t - i(k-\ell))^2 v + h_0 v = 0.$$

$$\begin{aligned}
\Phi(\ell) H \Phi^{-1}(\ell) &=: H_\ell + \ell 1\!\!1, \\
H_\ell &= \begin{pmatrix} 0 & 1\!\!1 \\ p(\ell) & 2(k-\ell) \end{pmatrix}, \ p(\ell) = h_0 - (k-\ell)^2.
\end{aligned}$$

2.4 Basic resolvent estimates and existence of the dynamics

Lemma (Basic resolvent estimates)

Let $\epsilon > 0$. We have

$$\begin{aligned} \|p^{-1}(z)u\| &\lesssim |z|^{-1}|\mathrm{Im} z|^{-1}\|u\|, \\ \|h_0^{1/2}p^{-1}(z)u\| &\lesssim |\mathrm{Im} z|^{-1}\|u\|. \end{aligned}$$

uniformly in $|z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}$, $|\mathrm{Im} z| > 0$.

Remark

i) Interpretation : superradiance does not occur for $|z| \geq (1+\epsilon)\|k\|$.
ii) Explanation : $p(z) = h_0 - (k-z)^2, \quad h_0 \geq 0$.

Lemma (Existence of the dynamics)

$(\dot{H}, D(\dot{H}))$ is the generator of a C_0- group $e^{-it\dot{H}}$ on $\dot{\mathcal{E}}$.

Definition

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{U}$ be a neighborhood of $z_0 \in \mathbb{C}$, and let $F : \mathcal{U} \setminus \{z_0\} \to \mathcal{B}(\mathcal{H})$ be a holomorphic function. F is finite meromorphic at z_0 if in the Laurent expansion $F(z) = \sum_{n=m}^{+\infty} (z - z_0)^n A_n, \quad m > -\infty$, the operators $A_m, ..., A_{-1}$ are of finite rank for $m < 0$. If in addition A_0 is a Fredholm operator, then F is called Fredholm at z_0.

Proposition

Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, let $Z \subset \mathcal{D}$ be a discrete and closed subset of $\mathcal{D}$, and let $F : \mathcal{D} \to \mathcal{B}(\mathcal{H})$ be a holomorphic function on $\mathcal{D} \setminus Z$. Assume that

- *F is finite meromorphic and Fredholm at each point of $\mathcal{D}$;*
- *there exists $z_0 \in \mathcal{D} \setminus Z$ such that $F(z_0)$ is invertible.*

Then there exists a discrete closed subset Z' of $\mathcal{D}$ such that $Z \subset Z'$ and :

- *$F(z)$ is invertible for $z \in \mathcal{D} \setminus Z'$;*
- *$F^{-1} : \mathcal{D} \setminus Z' \to \mathcal{B}(\mathcal{H})$ is finite meromorphic on $\mathcal{D}$ and Fredholm at each point of $\mathcal{D}$.*

3.2 Meromorphic extensions of weighted resolvents

Assumptions

(A3) $\quad h \geq 0, \quad 0 \notin \sigma_{pp}(h), \quad \forall u \in D(h^{1/2}),\ \|ku\| \lesssim \|h^{1/2}u\|.$

$(w, D(w))$ selfadjoint.
(ME1)

$$\begin{cases} a) & wkw \in \mathcal{B}(\mathcal{H}). \\ b) & [k, w] = 0. \\ c) & h^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2},\ [h, w^{-\epsilon}]w^{\epsilon/2}h^{-1/2},\ [h, w^{-\epsilon}]h^{-1/2} \in \mathcal{B}(\mathcal{H}),\ \forall \epsilon > 0 \\ d) & \forall \epsilon > 0,\ \|w^{-\epsilon}u\| \lesssim \|h^{1/2}u\|,\ \forall u \in h^{-1/2}\mathcal{H}, \\ e) & w^{-\epsilon}\langle h\rangle^{-\tilde{\epsilon}} \in \mathcal{B}_\infty(\mathcal{H}),\ \forall \epsilon, \tilde{\epsilon} > 0. \end{cases}$$

(ME2)
For all $\epsilon > 0$, $w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}$ extends from $\mathrm{Im} z > 0$ to $\mathrm{Im} z > -\delta_\epsilon$, $\delta_\epsilon > 0$

as a finite meromorphic function with values in $\mathcal{B}_\infty(\mathcal{H})$.

Proposition

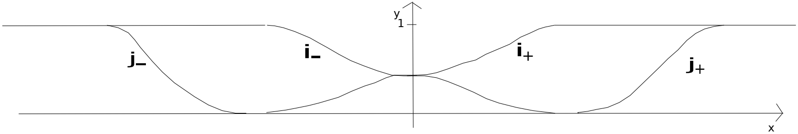
Assume (A1)-(A3), (ME1)-(ME2). Then $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ extends finite meromorphically to $\mathrm{Im} z > -\delta_{\epsilon/2}$ as an operator valued function with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$.

4 Klein-Gordon operators with "two ends"

4.1 Assumptions

$(x, D(x))$ selfadjoint, $\sigma(x) = \sigma_{ac}(x) = \mathbb{R}$, $[k, x] = 0$. Let $\chi_i \in C_b^\infty(\mathbb{R})$, $i = 1, 2$, $\quad \mathrm{supp}\,\chi_1 \cap \mathrm{supp}\,\chi_2 = \emptyset$. We suppose

(TE1)
$$\begin{cases} w = w(x), \quad w \in C^\infty(\mathbb{R}), \\ [x, k] = 0, \\ \chi_1(x) h_0 \chi_2(x) = 0. \end{cases}$$



$$\begin{cases} k_\pm &= k \mp \ell j_\mp^2, \\ h_\pm &= h_0 - k_\pm^2 \\ \tilde{h}_- &= h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2. \end{cases}$$

(TE2)
There exists $\ell \in \mathbb{R}$, $\epsilon > 0$ such that (h_+, k_+), $(\tilde{h}_-, k_- - \ell)$ satisfy (A3).

$$p_\pm(z) := h_\pm + z(2k_\pm - z).$$

Note that $\tilde{h}_- = p_-(\ell)$.

4.2 Asymptotic Hamiltonians

$$\dot{\mathcal{E}}_+ = h_+^{-1/2}\mathcal{H} \oplus \mathcal{H},\ \dot{\mathcal{E}}_- = \Phi(\ell)\tilde{h}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}.$$

$$\dot{H}_\pm = \begin{pmatrix} 0 & \mathbb{1} \\ h_\pm & 2k_\pm \end{pmatrix}.$$

are selfadjoint. We note $\dot{R}_\pm(z) := (\dot{H}_\pm - z)^{-1}$.
(TE3)

$$\begin{cases} a) & wi_+ki_+w,\ wi_-(k-\ell)i_-w \in \mathcal{B}(\mathcal{H}), \\ b) & [h, i_\pm] = \tilde{i}[h, i_\pm]\tilde{i}, \\ c) & (h_+, k_+, w) \quad \textit{and} \quad (\tilde{h}_-, k_- - \ell, w) \quad \text{fulfill (ME1), (ME2)}, \\ d) & h_\pm^{1/2} i_\pm h_\pm^{-1/2},\ h_0^{1/2} i_\pm h_0^{-1/2} \in \mathcal{B}(\mathcal{H}), \\ e) & w[h, i_\pm]wh_\pm^{-1/2},\ w[h, i_\pm]wh_0^{-1/2},\ [h, i_\pm]h_\pm^{-1/2}, \\ & [h, i_\pm]h_0^{-1/2},\ h_0^{-1/2}[w^{-1}, h_0]w \in \mathcal{B}(\mathcal{H}), \\ f) & (h_0, w) \quad \text{fulfill (ME1)d).} \end{cases}$$

Proposition

Let $\epsilon > 0$. Then $w^{-\epsilon}\dot{R}_\pm(z)w^{-\epsilon}$ extends finite meromorphically to $\mathrm{Im} z > -\delta_{\epsilon/2}$ as an operator valued function with values in $\mathcal{B}(\dot{\mathcal{E}}_\pm)$.

4.3 Construction of the resolvent

Proposition

If the conditions (A1)-(A2) and (TE1)-(TE3) are satisfied then there is a finite set $Z \subset \mathbb{C} \setminus \mathbb{R}$ with $\overline{Z} = Z$ such that the spectra of H and $\dot{H}$ are included in $\mathbb{R} \cup Z$ and such that the resolvents R and $\dot{R}$ are finite meromorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Moreover, the point spectrum of H coincides with the point spectrum of $\dot{H}$ and the set Z consists of eigenvalues of finite multiplicity of H and $\dot{H}$.

Proof.

$$Q(z) := i_-(\dot{H}_- - z)^{-1} i_- + i_+(\dot{H}_+ - z)^{-1} i_+.$$

$$\begin{aligned}
(\dot{H} - z)Q(z) &= \mathbb{1} + [\dot{H}, i_-](\dot{H}_- - z)^{-1} i_- + [\dot{H}, i_+](\dot{H}_+ - z)^{-1} i_+ \\
&= \mathbb{1} + K_-(z) + K_+(z) = 1 + K(z). \\
1 + K(z) &= (1 + K_-(z) j_- + K_+ j_+) \\
&\times (1 + K_-(z)(1 - j_-) + K_+(z)(1 - j_+)),
\end{aligned}$$

$$(1 + K_-(z) j_- + K_+(z) j_+)^{-1} = 1 - K_-(z) j_- - K_+(z) j_+.$$

Thus

$$\begin{aligned}
\dot{R}(z) &:= Q(z)(1 + K(z))^{-1} \\
&= Q(z)(1 + K_-(z)(1 - j_-) + K_+(1 - j_+))^{-1} \\
&\times (1 - K_-(z) j_- - K_+ j_+).
\end{aligned}$$

4.4 Smooth functional calculus

$$\|f\|_m := \sup_{\lambda\in\mathbb{R},\,\alpha\leq m} |f^{(\alpha)}(\lambda)|.$$

Proposition

Assume (A1), (A2), (TE1)-(TE3)*.*
(i) Let $f \in C_0^\infty(\mathbb{R})$*. Let* $\tilde{f}$ *be an almost analytic extension of* f *such that* $\operatorname{supp}\tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$*. Then the integral*
$f(\dot{H}) := \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(z)\dot{R}(z)dz\wedge d\bar{z}$
is norm convergent in $\mathcal{B}(\dot{\mathcal{E}})$ *and independent of the choice of the almost analytic extension of* f*.*
(ii) The map $C_0^\infty(\mathbb{R}) \ni f \mapsto f(\dot{H}) \in \mathcal{B}(\dot{\mathcal{E}})$ *is a homomorphism of algebras with*
$f(\dot{H})^* = \bar{f}(\dot{H}^*), \quad \|f(\dot{H})\|_{B(\dot{\mathcal{E}})} \leq \|f\|_m \quad$ *for some* $\quad m\in\mathbb{N}$*.*

Proposition

Assume $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$*. Let* $\chi \in C_0^\infty(\mathbb{R})$*,* $\chi\equiv 1$ *in a neighborhood of zero.*
Then $s-\lim_{L\to\infty}\chi\left(\frac{\dot{H}}{L}\right) = \mathbb{1}$*.*

5.1 Resonances and boundary values of the resolvent

Lemma

$w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ can be extended meromorphically from the upper half plane to $\mathrm{Im} z > -\delta_\epsilon,\ \delta_\epsilon > 0$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. poles: resonances.

We have $w^{-\epsilon}\dot{R}(z)w^{-\epsilon} = (\mathbb{1} + A_w(z))^{-1} w^{-\epsilon} Q(z) w^{-\epsilon}$.

Proposition

Assume (A1)-(A2), (TE1)-(TE3). Let $\epsilon > 0$. There exists a discrete closed set $\dot{\mathcal{T}}_H \subset \mathbb{R}$, $\nu > 0$ such that for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$ we have
(4)

$$\sup_{\|u\|_{\dot{\mathcal{E}}}=1,\ \nu \geq \delta > 0} \int_{\mathbb{R}} (\|w^{-\epsilon}\dot{R}(\lambda+i\delta)\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}} + \|w^{-\epsilon}\dot{R}(\lambda-i\delta)\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}})d\lambda < \infty.$$

Definition

We call $\lambda \in \mathbb{R}$ a regular point of $\dot{H}$ if there exists $\chi \in C_0^\infty(\mathbb{R})$, $\chi(\lambda) = 1$ such that (4) holds. Otherwise we call it a singular point.

Remark

Note that in the selfadjoint case $\dot{\mathcal{T}}_H$ is the set of real resonances by Kato's theory of H-smoothness.

5.2 Propagation estimates

Proposition

Assume (A1)-(A2), (TE1)-(TE3). Let $\epsilon > 0$. Then there exists a discrete closed set $\dot{\mathcal{T}} \subset \mathbb{R}$ such that for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}})$ and all $k \in \mathbb{N}$ we have

$$\|w^{-\epsilon} e^{-it\dot{H}} \chi(\dot{H}) w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \langle t \rangle^{-k}.$$

Proposition

Assume (A1)-(A2), (TE1)-(TE3). Let $\epsilon > 0$. Then we have for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$:

$$\int_{\mathbb{R}} \|w^{-\epsilon} e^{-it\dot{H}} \chi(\dot{H}) \varphi\|^2_{\dot{\mathcal{E}}} dt \lesssim \|\varphi\|^2_{\dot{\mathcal{E}}}.$$

Theorem

Suppose that $\lambda_0 \in \mathbb{R}$ is neither a resonance of $w^{-\epsilon} \dot{R}(\lambda) w^{-\epsilon}$ nor of $w^{-\epsilon} Q(\lambda) w^{-\epsilon}$. Then λ_0 is a regular point of $\dot{H}$.

Proof.

$$w^{-\epsilon} \dot{R}(z) = w^{-\epsilon} Q(z) - w^{-\epsilon} \dot{R}(z) w^{-\epsilon} w^{\epsilon} K(z).$$

□

6 Uniform boundedness of the evolution 1 : Abstract setting

(B1) $\left\{ \begin{array}{ll} a) & \text{For all } \psi \in C_0^\infty(\mathbb{R}), \quad h_0^{1/2}\psi(x)h_0^{-1/2} \in \mathcal{B}(\mathcal{H}). \\ b) & \text{If in addition } \psi \equiv 1 \text{ in a neighborhood of } 0,\ \psi \geq 0, \text{ then} \\ & s - \lim_{n\to\infty} \psi\left(\frac{x}{n}\right) = \mathbb{1} \quad \text{in} \quad h_0^{-1/2}\mathcal{H}. \end{array} \right.$

(B2) $[-ik, h] \lesssim w^{-1} h_0 w^{-1}$ in the sense of quadratic forms on $D(h_0)$.

For $\chi \in C^\infty(\mathbb{R})$ and $\mu > 0$ we put $\chi_\mu(.) = \chi\left(\frac{\cdot}{\mu}\right)$.

Theorem

Assume (A1), (A2), (TE1)-(TE3), (B1), (B2).
i) Let $\chi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi \subset \mathbb{R} \setminus [-1, 1]$, $\chi \equiv 1$ *on* $\mathbb{R} \setminus (-2, 2)$. *Then there exists* $\mu_0 > 0$, $C_1 > 0$ *such that we have for* $\mu \geq \mu_0$

$$\|e^{-it\dot{H}}\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_1\|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \quad \forall u \in \dot{\mathcal{E}}, \forall t \in \mathbb{R}.$$

ii) Let $\varphi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$. *Then there exists* $C_2 > 0$ *such that for all* $u \in \dot{\mathcal{E}}$ *and* $t \in \mathbb{R}$ *we have*

$$\|e^{-it\dot{H}}\varphi(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_2\|\varphi(\dot{H})u\|_{\dot{\mathcal{E}}}.$$

7 Asymptotic completeness 1 : Abstract setting

Definition

We call $\chi \in C^\infty(\mathbb{R})$ an admissible energy cut-off function for $\dot{H}$ if

- $\chi \equiv 0$ in a neighborhood of $\dot{\mathcal{T}}_H$ and
- $(\chi \equiv 0$ or $\chi \equiv 1)$ on $\mathbb{R} \setminus (-R, R)$ for some $R > 0$.

We note $\mathcal{C}^H$ the set of all admissible energy cut-offs for $\dot{H}$.

Definition

The spaces of scattering data are defined by
$\dot{\mathcal{E}}_{scatt} = \{\chi(\dot{H})u;\ u \in \dot{\mathcal{E}},\ \chi \in \mathcal{C}^H\}$,
$\dot{\mathcal{E}}^{\pm}_{scatt} = \{\chi(\dot{H}_\pm)u;\ u \in \dot{\mathcal{E}}_\pm,\ \chi \in \mathcal{C}^H\}$

Theorem

Assume (A1), (A2), (TE1)-(TE3), (B1)-(B2).
(i) For all $\varphi^\pm \in \dot{\mathcal{E}}^\pm_{scatt}$ *there exist* $\psi^\pm \in \dot{\mathcal{E}}_{scatt}$ *such that*

$$e^{-it\dot{H}}\psi^\pm - i_\pm e^{-it\dot{H}_\pm}\varphi^\pm \to 0,\ t \to \infty \quad in \quad \dot{\mathcal{E}}.$$

(ii) For all $\psi^\pm \in \dot{\mathcal{E}}_{scatt}$ *there exist* $\varphi^\pm \in \dot{\mathcal{E}}^\pm_{scatt}$ *such that*

$$e^{-it\dot{H}_\pm}\varphi^\pm - i_\pm e^{-it\dot{H}}\psi^\pm \to 0,\ t \to \infty \quad in \quad \dot{\mathcal{E}}_\pm.$$

8 Geometric setting

8.1 Separable hamiltonian

$\mathcal{M} = \mathbb{R}_{(r_-,r_+)} \times S_\omega^{d-1}$, $P = \sum_{ij=1}^{d-1} D_i^* \alpha_{ij} D_j \geq 0.$

$(G1)$ $L^2(S_\omega^{d-1}; d\omega) = \oplus_{n\in\mathbb{Z}} Y^n$, $D_{\theta_1}|_{Y^n} = n$, P leaves Y^n invariant.

$q(r) := \sqrt{(r_+ - r)(r - r_-)}$,

$T^\sigma = \{f \in C^\infty(\mathcal{M});\quad \partial_r^\alpha \partial_\omega^\beta f \in \mathcal{O}(q(r)^{\sigma-2\alpha})\}.$

$$h_0^s = \alpha_1 D_r \alpha_2^2 D_r \alpha_1 + \alpha_3^2 P + \alpha_4^2, \quad \alpha_i = \alpha_i(r).$$

(G2) $$\left\{ \begin{array}{rcll} \alpha_i - q(r)(i_-\alpha_i^- + i_+\alpha_i^+) & \in & T^{1+\delta}, & i = 1,2,3,4, \\ \alpha_i & \gtrsim & q(r), & i = 1,2,3,4. \end{array} \right.$$

$$k_s = k_{s,r} D_{\theta_1} + k_{s,v}.$$

(G3) $$\left\{ \begin{array}{rcl} i_+ k_{s,r},\ i_+ k_{s,v} & \in & T^2, \\ i_-(k_{s,r} - k_{s,r}^-) & \in & T^2, \\ i_-(k_{s,v} - k_{s,v}^-) & \in & T^2. \end{array} \right.$$

$\alpha_i^\pm$, $k_{s,r}^-$, $k_{s,v}^- \in \mathbb{R}$.

$h_s = h_0^s - k_s^2$, $\mathcal{H} = L^2(\mathbb{R}_{(r_-,r_+)} \times S_\omega^{d-1}; dr d\omega)$, $\mathcal{H}^n = \mathcal{H} \cap Y^n$.

8.2 Perturbed hamiltonian

$(\partial_t^2 - 2ik_s\partial_t + h_s)u = 0$. Perturbation: $(\partial_t^2 - 2ik\partial_t + h)u = 0$.

$$\begin{aligned} h_0|_{C_0^\infty(\mathcal{M})} &= h_0^s + \sum_{i,j\in\{1,\dots,d-1\}} D_i^* g^{ij} D_j + \sum_{i\in\{1,\dots,d-1\}} (g^i D_i + D_i^* \overline{g}i) \\ &+ D_r g^{rr} D_r + g^r D_r + D_r \overline{g}^r + f =: h_0^s + h_p. \end{aligned}$$

(G4) The functions g^{ij}, g^i, g^{rr}, g^r, f are independent of θ_1

(G5)
$$\begin{cases} h_0 & \gtrsim \quad q(r)(D_r q^2(r) D_r + P + 1)q(r), \\ h_0^s & \gtrsim \quad q(r)(D_r q^2(r) D_r + P + 1)q(r). \end{cases}$$

(G6)
$$\begin{cases} g^{ij} & \in \quad T^{2+\delta},\ i,j \in \{1,\dots,d-1\}, \\ g^{rr} & \in \quad T^{4+\delta} \\ g^r & \in \quad T^{2+\delta} \\ g^i & \in \quad T^{1+\delta},\ i \in \{1,\dots,d-1\} \\ f & \in \quad T^2. \end{cases}$$

$$k = k_r D_{\theta_1} + k_v,\ k_r = k_{s,r} + k_{p,r},\ k_v = k_{s,v} + k_{p,v}$$

(G7)
$$k_{p,v},\ k_{r,v} \in T^2.$$

$$h := h_0 - k^2.$$

9 Asymptotic completeness 2 : Geometric setting

$h_{+\infty} := h_0^s, \quad h_{-\infty} := h_{+\infty} - \ell^2,$

$k_{+\infty} := 0, \quad k_{-\infty} := \ell.$

Define $\dot{H}_{\pm\infty}$, $\dot{\mathcal{E}}_{\pm\infty}$, $\dot{\mathcal{E}}_{\pm\infty}$ in the usual manner.

Theorem

Assume (G1)-(G7).
(i) For all $\varphi^{\pm} \in \dot{\mathcal{E}}^{scatt}_{\pm\infty}$ there exist $\psi^{\pm} \in \dot{\mathcal{E}}^{scatt}$ such that

$$e^{-it\dot{H}}\psi^{\pm} - i_{\pm}e^{-it\dot{H}_{\pm\infty}}\varphi^{\pm} \to 0,\ t \to \infty \quad in \quad \dot{\mathcal{E}}.$$

(ii) For all $\psi^{\pm} \in \dot{\mathcal{E}}^{scatt}$ there exist $\varphi^{\pm} \in \dot{\mathcal{E}}^{scatt}_{\pm\infty}$ such that

$$e^{-it\dot{H}_{\pm\infty}}\varphi^{\pm} - i_{\pm}e^{-it\dot{H}}\psi^{\pm} \to 0,\ t \to \infty \quad in \quad \dot{\mathcal{E}}_{\pm\infty}.$$

Proposition

(G1)-(G7) entail (A1), (A2), (TE1)-(TE2), (B1), (B2).

Remark

h_+, $\tilde{h}_-$ are similar to the Laplacian on an asymptotically hyperbolic manifold. Meromorphic extension : Mazzeo-Melrose '87.

10.1 Uniform boundedness of the evolution

(5) $$\mathcal{H}^n = \{u \in L^2(\mathbb{R} \times S^2) : \ (D_\varphi - n)u = 0\}, \ n \in \mathbb{Z}.$$

We construct the energy spaces $\dot{\mathcal{E}}^n$, $\mathcal{E}^n$ as well as the Klein-Gordon operators H^n, $\dot{H}^n$ as in Sect. 1.

Theorem

There exists $a_0 > 0$ such that for $|a| < a_0$ the following holds: for all $n \in \mathbb{Z}$, there exists $C_n > 0$ such that

(6) $$\|e^{-it\dot{H}^n}u\|_{\dot{\mathcal{E}}^n} \le C_n \|u\|_{\dot{\mathcal{E}}^n}, \ u \in \dot{\mathcal{E}}^n, \ t \in \mathbb{R}.$$

Note that for $n = 0$ the Hamiltonian $\dot{H}^n = \dot{H}^0$ is selfadjoint, therefore the only issue is $n \neq 0$.

Proof :

Results of Dyatlov about the absence of complex eigenvalues and real resonances for $n \neq 0$, hypoellipticity argument, general result about the link between real resonances and singular points.

10.2 Asymptotic profiles

Let $\ell_\pm = \Omega_\pm n$. Also let $i_{l/r} \in C^\infty(\mathbb{R})$, $i_l = 0$ in a neighborhood of ∞, $i_r = 0$ in a neighborhood of $-\infty$ and $i_l^2 + i_r^2 = 1$. Let

$$h^n_{r/l} = -\partial_x^2 - \ell^2_{+/-},\ k_{r/l} = \ell_{+/-},$$

acting on $\mathcal{H}^n$ defined in (5).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{l/r}$ and Hamiltonians $\dot{H}^n_{l/r}$. Let $\{\lambda_q : q \in \mathbb{N}\} = \sigma(P)$ and $Z_q = \mathbb{1}_{\{\lambda_q\}}(P)\mathcal{H}$. Then

$$D(h_0) = D(h_{0,s}) = \{u \in \mathcal{H} : \sum_{q\in\mathbb{N}} \|h_0^{s,q}\mathbb{1}_{\{\lambda_q\}}(P)u\|^2 < \infty\},$$

where $h_0^{s,q}$ is the restriction of $h_{0,s}$ to $L^2(\mathbb{R}) \otimes Z_q$. Let

$$\begin{aligned} W_q &:= (L^2(\mathbb{R}) \otimes Z_q) \oplus (L^2(\mathbb{R}) \otimes Z_q),\ \mathcal{E}^{q,n}_{l/r} := \mathcal{E}^n_{r/l} \cap W_q, \\ \mathcal{E}^{fin,n}_{l/r} &:= \left\{ u \in \mathcal{E}^n_{l/r} : \exists Q > 0,\ u \in \oplus_{q\le Q}\mathcal{E}^{q,n}_{l/r} \right\}. \end{aligned}$$

Theorem

There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z} \setminus \{0\}$ the following holds:

- *i) For all $u \in \mathcal{E}^{fin,n}_{r/l}$ the limits*

$$W_{r/l}u = \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{r/l} e^{-it\dot{H}^n_{r/l}} u$$

  exist in $\dot{\mathcal{E}}^n$. The operators $W_{r/l}$ extend to bounded operators $W_{r/l} \in \mathcal{B}(\dot{\mathcal{E}}^n_{r/l}; \dot{\mathcal{E}}^n)$.

- *ii) The inverse wave operators*

$$\Omega_{r/l} = \mathrm{s}\text{-}\lim_{t\to\infty} e^{it\dot{H}^n_{r/l}} i^2_{r/l} e^{-it\dot{H}^n}$$

  exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{r/l})$.

i), ii) also hold for $n = 0$ if $m > 0$.

Remark

We can also compare to separable comparison dynamics for which the homogeneous energy spaces are the same.

Comments and Perspectives

- In the De Sitter-Kerr case it should be possible to get unifrom results in n using resolvent estimates that take into account the trapping (normally hyperbolic trapping). Work in progress.
- If a is "large" you can still do the gluing using more asymptotic hamiltonians, but the arguments of Dyatlov do not apply (perturbation arguments from the case $a = 0$).
- Asymptotic completeness results for hyperbolic equations on black hole spacetimes are necessary to give mathematically rigorous descriptions of the Hawking effect that predicts creation of particles by black holes.
- Because of the analytic extension of the resolvent we need a "very short range situation".
- If $\ell = 0$, i.e. k has the same limits in all the ends, the situation is much better because the conserved form is then continuous with respect to the natural norm. We can work with Krein spaces. In this setting we obtain a generalization of the Mourre theorem.

Theorem (Georgescu-Gérard-H 13)

Let $\mathcal{K}$ be a Krein space and A the generator of a C_0-group of operators on $\mathcal{K}$ such that the Krein structure is of class $C^1(A)$. Let H be a self-adjoint operator on $\mathcal{K}$ and Π a positive projection which commutes with H such that the following conditions are satisfied:

- *H is of class $C^\alpha(A)$ for some $\alpha > 3/2$, in particular $H' = [H, \mathrm{i}A]$ is well defined;*
- *there is $\varphi \in C_{\mathrm{c}}^\infty(\beta(H))$ real with $\varphi(\lambda) = 1$ on a neighborhood of a compact interval J such that $\varphi(H)\Pi = \varphi(H)$ and*

$$\varphi(H)(\mathrm{Re} H')\varphi(H) \geq a\varphi(H)^2, \; a > 0.$$

Then if $s > 1/2$ and $\varepsilon > 0$ is small enough, we have

$$\sup_{z \in J \pm \mathrm{i}]0,\nu]} \| \langle \varepsilon A \rangle^{-s} R(z) \langle \varepsilon A \rangle^{-s} \| < \infty, \text{ for some } \nu > 0.$$

Remark

To be able to apply the theorem one needs Borel type functional calculus. This can be obtained for so called definitizable operators. The Klein-Gordon equation coupled to an electromagnetic field enters into this setting

Thank you for your attention !