Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for random Schrödinger operators

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SPECTRAL DAYS 2014
CIRM (Luminy)
June 12, 2014
Schrödinger operators

We consider a Schrödinger operator

\[ H = -\Delta + V \quad \text{on} \quad L^2(\mathbb{R}^d), \]

where \( \Delta \) is the Laplacian operator and \( V \) is a bounded potential.
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- We define balls and boxes:
  \[ B(x, \delta) := \left\{ y \in \mathbb{R}^d; \|y - x\| < \delta \right\}, \quad \text{with} \quad \|x\| := \|x\|_2 = \left( \sum_{j=1}^{d} |x_j|^2 \right)^{\frac{1}{2}}; \]
  \[ \Lambda_L(x) := \left\{ y \in \mathbb{R}^d; \|y - x\|_\infty < \frac{L}{2} \right\}, \quad \text{with} \quad \|x\|_\infty := \max_{j=1,2,\ldots,d} |x_j|. \]
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\]

- \( H_\Lambda \) denotes the restriction of \( H \) to the the box \( \Lambda \subset \mathbb{R}^d \):

\[
H_\Lambda = -\Delta_\Lambda + V_\Lambda \quad \text{on} \quad L^2(\Lambda).
\]

- \( \Delta_\Lambda \) is the Laplacian on \( \Lambda \) with either Dirichlet or periodic boundary condition.

- \( V_\Lambda \) is the restriction of \( V \) to \( \Lambda \).
A UCPSP on a box $\Lambda$ is an estimate of the form

$$\chi_I(H_\Lambda) W_\Lambda \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda) \quad \text{on} \quad L^2(\Lambda),$$

where $\chi_I$ is the characteristic function of an interval $I \subset \mathbb{R}$, $W \geq 0$ is a potential, and $\kappa > 0$ is a constant.
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- If $W \geq \kappa > 0$ (covering condition) the UCPSP is trivial.
- Combes, Hislop and Klopp (2003): The UCPSP holds for bounded $\mathbb{Z}^d$-periodic potentials $V$ and $W$, $W \geq 0$ with $W > 0$ on an open set, boxes $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with $L \in \mathbb{N}$, $H_{\Lambda}$ with periodic boundary condition, with a constant $\kappa > 0$ depending on $\sup I$ (and $d, V, W$), but not on the box $\Lambda$. Their proof uses the unique continuation principle and Floquet theory.

Germinet and Klein (2013) proved a modified version of the CHK UCPSP, using Bourgain and Kenig’s quantitative unique continuation principle and (some) Floquet theory, obtaining control of the constant $\kappa$ in terms of the relevant parameters.
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- Given an energy $E_0 > 0$ and $\delta \in [0, \frac{1}{2}]$, define $\gamma = \gamma(d, K, \delta) > 0$ by

$$\gamma^2 = \frac{1}{2} \delta M_d \left(1 + K^2\right), \quad \text{where} \quad K = K(V, E_0) = 2 \|V\|_{\infty} + E_0.$$
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Where $\Lambda_1(k)$ is a box in $\mathbb{R}^d$. 

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we have
\[ \chi_I(H_\Lambda) W(\Lambda) \chi_I(H_\Lambda) \geq \gamma^2 \chi_I(H_\Lambda) \quad \text{on} \quad L^2(\Lambda). \]
Rojas-Molina and Veselić (2013) proved, under the hypotheses of the Theorem, that for boxes $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, if $\psi$ is an eigenfunction of $H_\Lambda$ with eigenvalue $E \in ]-\infty, E_0]$, then

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Our Theorem is derived from the quantitative unique continuation principle as in Bourgain and Klein using the “dominant boxes” introduced by Rojas-Molina and Veselić.
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Our Theorem is derived from the quantitative unique continuation principle as in Bourgain and Klein using the “dominant boxes” introduced by Rojas-Molina and Veselić.

The UCPSP is a crucial ingredient for proving Wegner estimates for one and multi-particle Anderson Hamiltonians. The UCPSP replaces the covering condition.
Quantitative unique cont. principle (Bourgain-Klein)

Let $\Omega \subset \mathbb{R}^d$ open. Let $\psi \in H^2(\Omega)$ and let $\zeta \in L^2(\Omega)$ be defined by

$$-\Delta \psi + V\psi = \zeta \quad \text{a.e. on } \Omega,$$

where $V$ is a bounded real measurable function on $\Omega$, $\|V\|_\infty \leq K < \infty$. 

Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi \chi_\Theta\|^2 > 0$.

Set $Q(x, \Theta) := \sup_{y \in \Theta} |y - x|$ for $x \in \Omega$.

Let $x_0 \in \Omega \setminus \Theta$ satisfy $Q = Q(x_0, \Theta) \geq 1$ and $B(x_0, 6Q + 2) \subset \Omega$.

Then, given $\delta > 0 \leq \min\{\text{dist}(x_0, \Theta), \frac{1}{2}\}$, we have

$$\left(\frac{\delta}{Q}\right)^m d \left(1 + \frac{K^2}{3}\right) \left(\frac{Q^4}{3} + \log \|\psi \chi_\Omega\|^2 \|\psi \chi_\Theta\|^2\right) \|\psi \chi_\Theta\|^2 \leq \|\psi \chi_{B(x_0, \delta)}\|^2 + \|\zeta \chi_\Omega\|^2,$$

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where $V$ is a bounded real measurable function on $\Omega$, $\|V\|_{\infty} \leq K < \infty$. Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi \chi_\Theta\|_2 > 0$. Then, given $0 < \delta \leq \min\{\text{dist}(x_0, \Theta), \frac{1}{2}\}$, we have

$$\left(\delta^4 + \frac{1}{3}\right)\left(\frac{1}{Q} + \log \frac{\|\psi \chi \|_2}{\|\psi \chi_\Theta\|_2^2}\right) \leq \|\psi \chi_{B(x_0, \delta)}\|_2^2 + \|\zeta \chi \|_2^2,$$

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we have

$$\left(\frac{\delta}{Q}\right)^{m_d (1 + K^\frac{2}{3})} \left( Q^\frac{4}{3} + \log \frac{\|\psi \chi_{\Omega}\|_2}{\|\psi \chi_{\Theta}\|_2} \right) \|\psi \chi_{\Theta}\|_2^2 \leq \|\psi \chi_{B(x_0, \delta)}\|_2^2 + \|\zeta \chi_{\Omega}\|_2^2,$$

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- Set $W^{(\Lambda)} = \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \chi_{B(y_k, \delta)}$. 

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Unique continuation principle for spectral projections

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- Consider a box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{R}^d$ and $L \geq 114 \sqrt{d}$.
- Set $W(\Lambda) = \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \chi_{B(y_k, \delta)}$.

Then for all real-valued $\psi \in \mathcal{D}(H_\Lambda) = \mathcal{D}(\Delta_\Lambda)$ we have (on $L^2(\Lambda)$)

\[ \|\psi\|_2^2 \leq M_d \left( \delta + K^2 \right) \left( \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \|\chi_B(y_k, \delta)\|_2^2 + \|H_\Lambda \psi\|_2^2 \right). \]
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\delta^{M_d \left(1 + K^2 \frac{2}{3}\right)} \|\psi\|_2^2 \leq \sum_{k \in \mathbb{Z}^d, \Lambda_1(k) \subset \Lambda} \|\psi \chi_{B(y_k, \delta)}\|_2^2 + \|H_\Lambda \psi\|_2^2
$$

$$
= \|W(\Lambda) \psi\|_2^2 + \|H_\Lambda \psi\|_2^2.
$$
Proof of the UCPSP

Let $E_0 > 0$ and $I = [E - \beta, E + \beta] \subset ]-\infty, E_0]$ a closed interval. Since $H_\Lambda \geq -\|V\|_\infty$, we assume $E \in [-\|V\|_\infty, E_0]$ without loss of generality, so

$$\|V - E\|_\infty \leq \|V\|_\infty + \max \{E_0, \|V\|_\infty\} \leq K = 2\|V\|_\infty + E_0.$$
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Moreover, for any box $\Lambda$ we have

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$$\delta^{M_d(1+K^2)} \|\psi\|_2^2 \leq \|W(\Lambda)\psi\|_2^2 + \|(H_\Lambda - E)\psi\|_2^2 \leq \|W(\Lambda)\psi\|_2^2 + \beta^2 \|\psi\|_2^2.$$
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$$\delta^d(\frac{1}{2}K^\frac{2}{3}) \|\psi\|_2^2 \leq \|W^{(\Lambda)}\psi\|_2^2 + \|(H_{\Lambda} - E)\psi\|_2^2 \leq \|W^{(\Lambda)}\psi\|_2^2 + \beta^2 \|\psi\|_2^2.$$  

If $\beta^2 \leq \gamma^2 := \frac{1}{2} \delta^d(\frac{1}{2}K^\frac{2}{3})$, i.e., $|I| \leq 2\gamma$, we get

$$\gamma^2 \|\psi\|_2^2 \leq \|W^{(\Lambda)}\psi\|_2^2, \quad \text{i.e.,} \quad \gamma^2 \chi_I(H_{\Lambda}) \leq \chi_I(H_{\Lambda}) W^{(\Lambda)} \chi_I(H_{\Lambda}).$$
Proof of the Corollary from the QUCP

Take $\Lambda = \Lambda_L(0)$ with $L \in \mathbb{N}_{\text{odd}}$. We extend functions $\varphi$ on $\Lambda$ to functions $\hat{\varphi}$ on $\mathbb{R}^d$ and $V$ to a potential $\hat{V}$ on $\mathbb{R}^d$ so

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Take $Y \in \mathbb{N}_{\text{odd}}$, $9 \leq Y < \frac{L}{2}$. Since $L$ is odd, we have $\Lambda = \bigcup_{k \in \Lambda \cap \mathbb{Z}^d} \Lambda_1(k)$. It follows that for all $\varphi \in L^2(\Lambda)$ we have

$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \tilde{\varphi}_\Lambda(k) \right\|_2^2 \leq (2Y)^d \left\| \varphi_\Lambda \right\|_2^2.$$
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$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \| \tilde{\varphi}_\Lambda(k) \|_2^2 \leq (2\gamma)^d \| \varphi_\Lambda \|_2^2.$$

We now fix $\psi \in \mathcal{D}(\Delta_\Lambda)$. Following Rojas-Molina and Veselić, we call a site $k \in \Lambda$ dominating (for $\psi$) if

$$\| \psi_{\Lambda_1(k)} \|_2^2 \geq \frac{1}{2(2\gamma)^d} \| \tilde{\psi}_{\Lambda_\gamma(k)} \|_2^2.$$
Proof of the Corollary from the QUCP

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Take $Y \in \mathbb{N}_{\text{odd}}$, $9 \leq Y < \frac{L}{2}$. Since $L$ is odd, we have $\bar{\Lambda} = \bigcup_{k \in \Lambda \cap \mathbb{Z}^d} \Lambda_1(k)$. It follows that for all $\varphi \in L^2(\Lambda)$ we have

$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \| \tilde{\varphi}_{\Lambda Y}(k) \|_2^2 \leq (2Y)^d \| \varphi_\Lambda \|_2^2.$$

We now fix $\psi \in \mathcal{D}(\Delta_\Lambda)$. Following Rojas-Molina and Veselić, we call a site $k \in \hat{\Lambda}$ dominating (for $\psi$) if

$$\| \psi_{\Lambda 1}(k) \|_2^2 \geq \frac{1}{2(2Y)^d} \| \tilde{\psi}_{\Lambda Y}(k) \|_2^2,$$

and note that, letting $\hat{D} \subset \Lambda \cap \mathbb{Z}^d$ denote the collection of dominating sites,

$$\sum_{k \in \hat{D}} \| \psi_{\Lambda 1}(k) \|_2^2 \geq \frac{1}{2} \| \psi_\Lambda \|_2^2.$$
Proof of the Corollary-continued

If $k \in \hat{D}$ we apply the QUCP with $\Omega = \Lambda_Y(k)$ and $\Theta = \Lambda_1(k)$, obtaining

$$
\delta^m d \left(1 + K^2 \right) \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \leq \left\| \psi_{B(y_J(k), \delta)} \right\|_2^2 + \left\| \tilde{\zeta}_{\Lambda Y(k)} \right\|_2^2,
$$

where $\tilde{\zeta} = (-\Delta + V) \psi$, $Y$ is appropriately chosen, $Y \leq 40 \sqrt{d} < L^2$, and the map $J: \hat{D} \to \Lambda \cap \mathbb{Z}^d$ is defined appropriately so $J(k) \in \Lambda_Y(k)$ and $\# J^{-1}(\{ j \}) \leq 2$ for all $j$. Summing over $k \in \hat{D}$ and using $\sum_{k \in \hat{D}} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \geq \frac{1}{2} \left\| \psi_{\Lambda} \right\|_2^2$, we get

$$
\frac{1}{2} \delta^m d \left(1 + K^2 \right) \left\| \psi_{\Lambda} \right\|_2^2 \leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \psi_{B(y_J(k), \delta)} \right\|_2^2 + (2 Y d) \left\| \tilde{\zeta}_{\Lambda Y(k)} \right\|_2^2,
$$

which implies (with a different constant $M_d > 0$)

$$
\delta^m d \left(1 + K^2 \right) \left\| \psi_{\Lambda} \right\|_2^2 \leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \psi_{\chi_{B(y_J(k), \delta)}} \right\|_2^2 + \left\| \tilde{\zeta}_{\Lambda Y(k)} \right\|_2^2.
$$
Proof of the Corollary-continued

If \( k \in \hat{D} \) we apply the QUCP with \( \Omega = \Lambda_Y(k) \) and \( \Theta = \Lambda_1(k) \), obtaining

\[
\delta^{m'_d (1 + \kappa^{\frac{2}{3}})} \| \psi_{\Lambda_1(k)} \|_2^2 \leq \| \psi_{B(\gamma_{\mathcal{J}(k)}, \delta)} \|_2^2 + \| \tilde{\zeta}_{\Lambda_Y(k)} \|_2^2,
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where \( \zeta = (\Delta + V)\psi \), \( Y \) is appropriately chosen, \( Y \leq 40 \sqrt{d} < \frac{L}{2} \), and the map \( J: \hat{D} \to \Lambda \cap \mathbb{Z}^d \) is defined appropriately so \( J(k) \in \Lambda_Y(k) \) and \( \# J^{-1}(\{j\}) \leq 2 \) for all \( j \).
If \( k \in \hat{D} \) we apply the QUCP with \( \Omega = \Lambda_Y(k) \) and \( \Theta = \Lambda_1(k) \), obtaining

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Summing over \( k \in \hat{D} \) and using \( \sum_{k \in \hat{D}} \| \psi_{\Lambda_1(k)} \|_2^2 \geq \frac{1}{2} \| \psi_{\Lambda} \|_2^2 \), we get

\[
\frac{1}{2} \delta^m_d \big( 1 + K^3 \big) \| \psi_{\Lambda} \|_2^2 \leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \| \psi_{B(y_k, \delta)} \|_2^2 + (2Y)^d \| \zeta_{\Lambda} \|_2^2
\]

\[
\leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \| \psi_{B(y_k, \delta)} \|_2^2 + (80\sqrt{d})^d \| \zeta_{\Lambda} \|_2^2,
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If \( k \in \hat{D} \) we apply the QUCP with \( \Omega = \Lambda_Y(k) \) and \( \Theta = \Lambda_1(k) \), obtaining

\[
\delta^m \left(1 + K^2 \right) \| \psi \|_2^2 \leq \| \psi_B(y_J(k), \delta) \|_2^2 + \| \tilde{\zeta} \psi \Lambda_Y(k) \|_2^2,
\]

where \( \zeta = (-\Delta + V)\psi \), \( Y \) is appropriately chosen, \( Y \leq 40\sqrt{d} < \frac{L}{2} \), and the map \( J: \hat{D} \to \Lambda \cap \mathbb{Z}^d \) is defined appropriately so

\[
J(k) \in \Lambda_Y(k) \quad \text{and} \quad \#J^{-1}(\{j\}) \leq 2 \quad \text{for all } j.
\]

Summing over \( k \in \hat{D} \) and using \( \sum_{k \in \hat{D}} \| \psi \Lambda_Y(k) \|_2^2 \geq \frac{1}{2} \| \psi \Lambda \|_2^2 \), we get

\[
\frac{1}{2} \delta^m \left(1 + K^2 \right) \| \psi \Lambda \|_2^2 \leq 2 \sum_{k \in \Lambda \cap \mathbb{Z}^d} \| \psi_B(y_k, \delta) \|_2^2 + (2Y)^d \| \zeta \Lambda \|_2^2
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A crooked Anderson Hamiltonian is the random Schrödinger operator

\[ H_\omega := H_0 + V_\omega \quad \text{on} \quad L^2(\mathbb{R}^d) \]
Crooked Anderson Hamiltonians

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1. \( H_0 = -\Delta + V^{(0)} \), with \( V^{(0)} \) a bounded potential and \( \inf \sigma(H_0) = 0 \).

Remark: If \( V^{(0)} \) is \( \mathbb{Z}^d \)-periodic with \( q \in \mathbb{N} \), and \( y_j = j \), \( v_j = v_0 \), \( \mu_j = \mu_0 \) for all \( j \in \mathbb{Z}^d \), then \( H_\omega \) is the ergodic (usual) Anderson Hamiltonian.
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1. $H_0 = -\Delta + V^{(0)}$, with $V^{(0)}$ a bounded potential and $\inf \sigma(H_0) = 0$.
2. $V_\omega$ is a crooked alloy-type random potential:

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x), \text{ with } u_j(x) = v_j(x - y_j),$$

where, for some $\delta_- \in ]0, \frac{1}{2}]$ and $u_-, \delta_+, M \in ]0, \infty[$.
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1. \( \{y_j\}_{j \in \mathbb{Z}^d} \) are sites in \( \mathbb{R}^d \) with \( B(y_j, \delta_-) \subset \Lambda_1(j) \) for all \( j \in \mathbb{Z}^d \);
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Finite volume crooked Anderson Hamiltonians

We define finite volume crooked Anderson Hamiltonians on a box
$\Lambda = \Lambda_L(x_0), \ x_0 \in \mathbb{R}^d$ and $L > 0$, with either Dirichlet or periodic boundary condition, by

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We also set

\[
U(x) := \sum_{j \in \mathbb{Z}^d} u_j(x) \quad \text{and} \quad U^{(\Lambda)}(x) := \sum_{j \in \Lambda \cap \mathbb{Z}^d} u_j(x),
\]

\[
W(x) := \sum_{j \in \mathbb{Z}^d} \chi_{B(y_j,\delta)}(x) \quad \text{and} \quad W^{(\Lambda)}(x) := \sum_{j \in \mathbb{Z}^d, \Lambda_1(j) \subset \Lambda} \chi_{B(y_j,\delta)}.
\]
Remark and notation

Note that

\[ 0 \leq W_\Lambda \leq \frac{1}{u_-} U_\Lambda. \]
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S_\mu(t) := \sup_{a \in \mathbb{R}} \mu([a, a + t]) \quad \text{for} \quad t \geq 0.
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- \( S_\Lambda(t) := \max_{j \in \Lambda \cap \mathbb{Z}^d} S_{\mu_j}(t). \)
Optimal Wegner estimates

An optimal Wegner estimate for Anderson Hamiltonians is an estimate of the form

$$\mathbb{E}\{ \text{tr} P_{\omega,\Lambda}(I) \} \leq C S_\Lambda(|I|)|\Lambda|.$$
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- Combes, Hislop, Klopp (2007) proved optimal Wegner estimates for ergodic Anderson Hamiltonians with periodic boundary condition and boxes \( \Lambda = \Lambda_L(x_0) \) with \( L \) a multiple of the period.
  Their proof uses the UCSP for the (nonrandom) periodic operator \( H_0 \).
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- Combes, Hislop, Klopp (2007) proved optimal Wegner estimates for ergodic Anderson Hamiltonians with periodic boundary condition and boxes $\Lambda = \Lambda_L(x_0)$ with $L$ a multiple of the period. Their proof uses the UCSP for the (nonrandom) periodic operator $H_0$.
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They used their single energy UCSP for the (nonrandom) operator $H_0$. 

Wegner estimates for crooked Anderson Hamiltonians imply corresponding Wegner estimates for Delone-Anderson models.
Optimal Wegner estimates

An optimal Wegner estimate for Anderson Hamiltonians is an estimate of the form

\[ E \left\{ \text{tr} \, P_{\omega, \Lambda}(I) \right\} \leq C \, S_{\Lambda}(|I|) |\Lambda|. \]

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Optimal Wegner estimate for crooked Anderson Hamiltonians.

Using the UCPSP for the full random operator $H_\omega$, we prove

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Let $H_\omega$ be a crooked Anderson Hamiltonian. Given $E_0 > 0$, define $\gamma > 0$ by

$$\gamma^2 = \frac{1}{2} \delta_-^{M_d (1 + K^{2/3})}, \quad \text{where} \quad K = E_0 + 2 \left( \| V^{(0)} \|_\infty + M \| U \|_\infty \right).$$

and $M_d > 0$ is the constant in the UCPSP Theorem.
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Then for any closed interval $I \subset ]-\infty, E_0]$ with $|I| \leq 2\gamma$ and any box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{R}^d$ and $L \geq 114\sqrt{d} + \delta_+$,
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$$\mathbb{E} \left\{ \text{tr} \ P_{\omega,\Lambda}(I) \right\} \leq C_{d,\delta_+} \| V^{(0)} \|_\infty \left(u^{-2} \gamma^{-4} (1 + E_0)\right)^{2 + \frac{\log d}{\log 2}} S_\Lambda(|I|) |\Lambda|.$$

Abel Klein

Unique continuation principle for spectral projections
UCPSP $\implies$ Optimal Wegner estimate

The theorem (optimal Wegner estimates) follows from the UCPSP theorem and the following lemma.
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**Lemma**

Let $H_\omega$ be a crooked Anderson Hamiltonian. Let $I \subset ]-\infty, E_0]$ be a closed interval and $\Lambda = \Lambda_L(x_0)$ a box centered at $x_0 \in \mathbb{R}^d$ with $L \geq 2 + \delta_+$. 

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Suppose there exists a constant $\kappa > 0$ such that

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**Lemma**

Let $H_\omega$ be a crooked Anderson Hamiltonian. Let $I \subset [-\infty, E_0]$ be a closed interval and $\Lambda = \Lambda_L(x_0)$ a box centered at $x_0 \in \mathbb{R}^d$ with $L \geq 2 + \delta_+$. Suppose there exists a constant $\kappa > 0$ such that

$$P_{\omega, \Lambda}(I) U^{(\Lambda)} P_{\omega, \Lambda}(I) \geq \kappa P_{\omega, \Lambda}(I) \quad \text{with probability one.}$$

Then

$$\mathbb{E} \left\{ \text{tr} \; P_{\omega, \Lambda}(I) \right\} \leq C_{d, \delta_+, \|V^{(0)}\|_\infty} \left( \frac{\kappa^{-2}}{(1 + E_0)} \right)^{2 \left( 1 + \frac{\log d}{\log 2} \right)} S_{\Lambda}(|I|) |\Lambda|. $$
Proof of Lemma

We fix $\Lambda$ and $I \subset ]-\infty, E_0]$, let $P = P_{\omega,\Lambda}(I)$, $U = U^{(\Lambda)}$. Then (Dirichlet bc)
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$$= \kappa^{-2} \text{tr } PUPUP \leq \kappa^{-2} (1 + E_0) \text{tr } PU(H_{\omega, \Lambda} + 1)^{-1} UP$$

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where $T_{ij} = \sqrt{u_i}(H_{0, \Lambda} + 1)^{-1} \sqrt{u_j}$ for $i,j \in \Lambda \cap \mathbb{Z}^d$. 

Abel Klein  Unique continuation principle for spectral projections
Proof of Lemma

We fix $\Lambda$ and $I \subset [-\infty, E_0]$, let $P = P_{\omega, \Lambda}(I)$, $U = U^{(\Lambda)}$. Then (Dirichlet bc)

$$\text{tr } P \leq \kappa^{-1} \text{tr } PUP = \kappa^{-1} \text{tr } \sqrt{UP}\sqrt{U} \leq \kappa^{-2} \text{tr } \sqrt{UPUP}\sqrt{U} = \kappa^{-2} \text{tr } PUPU$$

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$$= \kappa^{-2}(1 + E_0) \sum_{i,j \in \Lambda \cap \mathbb{Z}^d} \text{tr } \sqrt{u_j P}\sqrt{u_i} T_{ij},$$

where $T_{ij} = \sqrt{u_i}(H_{0, \Lambda} + 1)^{-1}\sqrt{u_j}$ for $i,j \in \Lambda \cap \mathbb{Z}^d$.

We may now adapt an argument in in Combes, Hislop, Klopp obtaining

$$\mathbb{E} \text{tr } P \leq C_{d, \delta_+, \nu_\infty^{(0)}} \left( \kappa^{-2}(1 + E_0) \right)^{2 + \frac{\log d}{\log 2}} S_{\Lambda}(|I|)|\Lambda|.$$
Multi-particle Anderson Hamiltonians

The $n$-particle Anderson Hamiltonian is the random Schrödinger operator

$$H^{(n)}_\omega := H^{(n)}_{0,\omega} + U \quad \text{on} \quad \mathbf{L}^2(\mathbb{R}^{nd}),$$

where

$$H^{(n)}_{0,\omega} := -\Delta^{(n)} + V^{(n)}_\omega.$$

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Abel Klein

Unique continuation principle for spectral projections
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1. \( \omega = \{\omega_k\}_{k \in \mathbb{Z}^d} \) is a family of independent identically distributed random variables whose common probability distribution \( \mu \) has a bounded density \( \rho \) and satisfies \( \{0, M_+\} \subset \text{supp} \mu \subset [0, M_+] \) for some \( M_+ > 0 \).
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1. $\omega = \{\omega_k\}_{k \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables whose common probability distribution $\mu$ has a bounded density $\rho$ and satisfies $\{0, M_+\} \subset \text{supp} \mu \subset [0, M_+]$ for some $M_+ > 0$;
2. the single site potential $u$ is a measurable function on $\mathbb{R}^d$ with

$$u - \chi_{\Lambda_-(0)} \leq u \leq \chi_{\Lambda_+(0)}, \quad \text{where} \quad u_-, \delta \in (0, \infty), \quad \Lambda_\delta(0) = (-\frac{\delta}{2}, \frac{\delta}{2})^d.$$
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V^{(n)}_{\omega}(\mathbf{x}) = \sum_{i=1}^{n} V^{(1)}_{\omega}(x_i), \quad \text{with} \quad V^{(1)}_{\omega}(\mathbf{x}) = \sum_{k \in \mathbb{Z}^d} \omega_k u(\mathbf{x} - k),
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\]
3. \( U \) is a short range interaction potential between the \( n \) particles:

\[
U(\mathbf{x}) = \sum_{1 \leq i < j \leq n} \tilde{U}(x_i - x_j),
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\[
0 \leq \tilde{U}(y) \leq \tilde{U}_\infty < \infty, \quad \tilde{U}(y) = \tilde{U}(-y), \quad \tilde{U}(y) = 0 \quad \text{for} \quad \|y\|_\infty > r_0 \in (0, \infty).
\]
Notation

1. Given \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we set \( \| x \| = \| x \|_\infty := \max \{ |x_1|, \ldots, |x_d| \} \).
Notation

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Notation

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3. The one-particle box centered at $x \in \mathbb{R}^d$ with side of length $L > 0$ is $\Lambda_L(x) = \{y \in \mathbb{R}^d; \|y - x\| < \frac{L}{2}\}$. We set $\hat{\Lambda} = \Lambda \cap \mathbb{Z}^d$. 
Optimal Wegner estimates – multi-particles

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4. The \( n \)-particle box centered at \( x \in \mathbb{R}^{nd} \) with side length \( L > 0 \) is

\[
\Lambda_L^{(n)}(x) = \left\{ y \in \mathbb{R}^{nd}; \|y - x\| < \frac{L}{2}\right\} = \prod_{i=1}^{n} \Lambda_L(x_i);
\]

note that \( \Lambda_L^{(1)}(x) = \Lambda_L(x) \). By a box \( \Lambda_L \) in \( \mathbb{R}^{nd} \) we mean an \( n \)-particle box \( \Lambda_L^{(n)}(x) \) for some \( x \in \mathbb{R}^{nd} \).
Given $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we set $\|x\| = \|x\|_\infty := \max\{|x_1|, \ldots, |x_d|\}$.

If $a = (a_1, \ldots, a_n) \in \mathbb{R}^{nd}$, we set $\|a\| := \max\{\|a_1\|, \ldots, \|a_n\|\}$.

The one-particle box centered at $x \in \mathbb{R}^d$ with side of length $L > 0$ is $\Lambda_L(x) = \{y \in \mathbb{R}^d; \|y - x\| < \frac{L}{2}\}$. We set $\hat{\Lambda} = \Lambda \cap \mathbb{Z}^d$.

The $n$-particle box centered at $x \in \mathbb{R}^{nd}$ with side length $L > 0$ is

$$\Lambda_L^{(n)}(x) = \left\{ y \in \mathbb{R}^{nd}; \|y - x\| < \frac{L}{2}\right\} = \prod_{i=1}^{n} \Lambda_L(x_i);$$

note that $\Lambda_L^{(1)}(x) = \Lambda_L(x)$. By a box $\Lambda_L$ in $\mathbb{R}^{nd}$ we mean an $n$-particle box $\Lambda_L^{(n)}(x)$ for some $x \in \mathbb{R}^{nd}$.

Given a one-particle box $\Lambda$, we will use $E_{\Lambda}$ and $P_{\Lambda}$ to denote the expectation and probability with respect to the probability distribution of the random variables $\{\omega_k\}_{k \in \hat{\Lambda}}$. 
Finite volume multi-particle Anderson Hamiltonians

Given an $n$-particle box $\Lambda = \Lambda^{(n)}_L(a)$, we define the corresponding finite volume Anderson Hamiltonian $H^{(n)}_{\omega,\Lambda}$ on $L^2(\Lambda)$ by

$$H^{(n)}_{\omega,\Lambda} := H^{(n)}_{0,\omega,\Lambda} + U_{\Lambda},$$

with $H^{(n)}_{0,\omega,\Lambda} := -\Delta^{(n)}_{\Lambda} + V^{(n)}_{\omega,\Lambda}$.
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where $\Delta^{(n)}_{\Lambda}$ is the Laplacian on $\Lambda$ with Dirichlet boundary condition, $U_{\Lambda}$ is the restriction of $U$ to $\Lambda$, $V^{(n)}_{\omega,\Lambda}(x) = \sum_{k \in \hat{\Lambda}_{\omega}} k u(x - k)$ for $x \in \Lambda$. 

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Unique continuation principle for spectral projections
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Given an $n$-particle box $\Lambda = \Lambda^{(n)}_{L}(a)$, we define the corresponding finite volume Anderson Hamiltonian $H^{(n)}_{\omega, \Lambda}$ on $L^{2}(\Lambda)$ by

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where $\Delta^{(n)}_{\Lambda}$ is the Laplacian on $\Lambda$ with Dirichlet boundary condition, $U_{\Lambda}$ is the restriction of $U$ to $\Lambda$, and

$$V^{(n)}_{\omega, \Lambda}(x) = \sum_{i=1}^{n} V^{(1)}_{\omega, \Lambda}(a_{i})(x_{i}) \quad \text{for} \quad x \in \Lambda,$$

where $V^{(1)}_{\omega, \Lambda}$ is defined for a one-particle box $\Lambda \subseteq \mathbb{R}^{d}$ by

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H^{(n)}_{\omega,\Lambda} := H^{(n)}_{0,\omega,\Lambda} + U_\Lambda,
\]

with \( H^{(n)}_{0,\omega,\Lambda} := -\Delta^{(n)}_\Lambda + V^{(n)}_{\omega,\Lambda} \),

where \( \Delta^{(n)}_\Lambda \) is the Laplacian on \( \Lambda \) with Dirichlet boundary condition, \( U_\Lambda \) is the restriction of \( U \) to \( \Lambda \), and

\[
V^{(n)}_{\omega,\Lambda}(x) = \sum_{i=1}^n V^{(1)}_{\omega,\Lambda_L(a_i)}(x_i) \quad \text{for} \quad x \in \Lambda,
\]

where \( V^{(1)}_{\omega,\Lambda} \) is defined for a one-particle box \( \Lambda \subseteq \mathbb{R}^d \) by

\[
V^{(1)}_{\omega,\Lambda}(x) = \sum_{k \in \hat{\Lambda}} \omega_k \, u(x - k) \quad \text{for} \quad x \in \Lambda.
\]

We set

\[
R^{(n)}_{\omega,\Lambda}(z) = (H^{(n)}_{\omega,\Lambda} - z)^{-1} \quad \text{for} \quad z \notin \sigma(H^{(n)}_{\omega,\Lambda}).
\]
Wegner estimate for multi-particle Anderson Hamiltonians

Theorem

Let $n \in \mathbb{N}$ and $E_+ > 0$. There exist constants $\gamma_{n,E_+} > 0$ and $C_{n,E_+}$, such that, for all $n$-particle boxes $\Lambda = \Lambda^{(n)}_L(a)$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^{nd}$ and $L \geq 114 \sqrt{nd}$ and all intervals $I \subseteq [0, E_+]$ with $|I| \leq 2 \gamma_{n,E_+}$, we have

$$
\mathbb{E}_{\Lambda_L(a_i)} \left\{ \text{tr} \chi_I \left( H^{(n)}_{\omega, \Lambda} \right) \right\} \leq C_{n,E_+} \| \rho \|_\infty |I| L^{nd} \quad \text{for} \quad i = 1, 2, \ldots, n.
$$

In particular, for any $E \leq E_+$, $0 < \varepsilon \leq \gamma_{n,E_+}$, and $i = 1, 2, \ldots, n$, we have

$$
\mathbb{P}_{\Lambda_L(a_i)} \left\{ d(\sigma(H^{(n)}_{\omega, \Lambda}), E) \leq \varepsilon \right\} \leq 2 C_{n,E_+} \| \rho \|_\infty \varepsilon L^{nd}.
$$
Wegner estimate for multi-particle Anderson Hamiltonians

Theorem

Let $n \in \mathbb{N}$ and $E_+ > 0$. There exist constants $\gamma_{n,E_+} > 0$ and $C_{n,E_+}$, such that, for all $n$-particle boxes $\Lambda = \Lambda_L^{(n)}(a) \text{ with } a = (a_1, \ldots, a_n) \in \mathbb{R}^{nd}$ and $L \geq 114 \sqrt{nd}$ and all intervals $I \subseteq [0, E_+)$ with $|I| \leq 2 \gamma_{n,E_+}$, we have

$$\mathbb{E}_{\Lambda_L(a_i)} \left\{ \text{tr} \chi_I \left( H^{(n)}_{\omega, \Lambda} \right) \right\} \leq C_{n,E_+} \| \rho \|_{\infty} |I| L^{nd} \quad \text{for } i = 1, 2, \ldots, n.$$ 

In particular, for any $E \leq E_+$, $0 < \varepsilon \leq \gamma_{n,E_+}$, and $i = 1, 2, \ldots, n$, we have

$$\mathbb{P}_{\Lambda_L(a_i)} \left\{ d(\sigma(H^{(n)}_{\omega, \Lambda}), E) \leq \varepsilon \right\} \leq 2C_{n,E_+} \| \rho \|_{\infty} \varepsilon L^{nd}.$$ 

Hislop and Klopp: similar Wegner estimate taking expectation over all random variables.
Proof of multi-particle Wegner estimate

Let $\Lambda = \Lambda_L(n)(a)$, $\Lambda_i = \Lambda_L(a_i)$.

$$V_{\omega, \Lambda}(x) = \sum_{i=1}^{n} V_{\omega, \Lambda_i}(x_i) = \sum_{i=1}^{n} \sum_{k \in \hat{\Lambda}_i} \omega_k u(x_i - k) = \sum_{k \in \mathbb{Z}^d} \omega_k \theta_k^{(\Lambda)}(x),$$

$$\theta_k^{(\Lambda)}(x) = \sum_{\{i; k \in \hat{\Lambda}_i\}} u(x_i - k) \geq u_{-} \sum_{\{i; k \in \hat{\Lambda}_i\}} \chi_{\Lambda_1}^{(1)}(x_i).$$

Fix $q \in \{1, 2, \ldots, n\}$, we have

$$H_{\omega, \Lambda}^{(n)} = -\Delta^{(n)} + U_{\Lambda} + \sum_{k \in \mathbb{Z}^d \setminus \hat{\Lambda}_q} \omega_k \theta_k^{(\Lambda)} + \sum_{k \in \hat{\Lambda}_q} \omega_k \theta_k^{(\Lambda)}.$$

Then for $x \in \Lambda$ we have (with $\eta = \min\{\frac{\delta_-}{2}, \frac{1}{2}\}$)

$$W^{(\Lambda)}(x) := \sum_{k \in \Lambda \cap \mathbb{Z}^{nd}} \chi_{B^{(n)}(k, \eta)}(x) \leq u_{-1}^{-1} \sum_{k \in \hat{\Lambda}_q} \theta_k^{(\Lambda)}(x).$$
Fix $E_+ > 0$. It follows from the UCPSP Theorem that for any interval $I \subseteq [0, E_+]$ with $|I| \leq 2\gamma_{n,E_+}$ we have

$$\chi_I(H_{\omega,\Lambda}^{(n)}) \leq \gamma_{n,E_+}^{-2} \chi_I(H_{\omega,\Lambda}^{(n)}) W(\Lambda) \chi_I(H_{\omega,\Lambda}^{(n)})$$

$$\leq u_{-1}^{-1} \gamma_{n,E_+}^{-2} \chi_I(H_{\omega,\Lambda}^{(n)}) \left( \sum_{k \in \tilde{\Lambda}_q} \theta_k^{(\Lambda)} \right) \chi_I(H_{\omega,\Lambda}^{(n)}),$$

where $\gamma_{n,E_+}^2 = \frac{1}{2} \eta^{M_{nd}} (1 + K^{\frac{2}{3}})$ with $K = n(n - 1) \|\tilde{U}\|_\infty + 2M_+ \delta_+ + E_+$.

The Wegner estimate can now be proved following as in one-particle case, averaging only the random variables $\{\omega_i\}_{i \in \tilde{\Lambda}_q}$. 
Let $H_{\omega, \lambda} = H_0 + \lambda V_\omega$ be a crooked Anderson Hamiltonian, where $\lambda > 0$ is the disorder parameter.
Wegner estimates at high disorder

Let $H_{\omega,\lambda} = H_0 + \lambda V_\omega$ be a crooked Anderson Hamiltonian, where $\lambda > 0$ is the disorder parameter. We can make explicit the dependence on $\lambda$ in the Wegner estimate:

$$\mathbb{E} \left\{ \text{tr} \ P_{\omega,\lambda,\Lambda}(I) \right\} \leq C E_0 e^{c E_0 \left(1 + \lambda^{2\frac{3}{2}} \right)} S_{\Lambda}(\lambda^{-1} |I| |\Lambda|).$$
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If we use the UCPSP for $H_0$, as in Combes, Hislop and Klopp, we get

$$\mathbb{E} \{ \text{tr} P_{\omega,\lambda}(I) \} \leq C_{E_0} \left(1 + \lambda^{2 + \frac{\log d}{\log 2}}\right) S_\lambda (\lambda^{-1} |I|) |\Lambda|.$$
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These Wegner estimates get worse as the disorder increases.
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But if we have the covering condition $U^{(\Lambda)} \geq \alpha \chi_\Lambda$ for some $\alpha > 0$, we get, following Combes-Hislop or the Lemma,

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\mathbb{E} \left\{ \text{tr} P_{\omega,\lambda,\Lambda}(I) \right\} \leq C_{d,\delta,+},\alpha,\|V^{(0)}\|_\infty, E_0 S_\Lambda(\lambda^{-1} |I|) |\Lambda|.
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$$\mathbb{E} \left\{ \text{tr} \, P_{\omega, \lambda, \Lambda}(I) \right\} \leq C_{d, \delta_+, \alpha, \|V^{(0)}\|_\infty, E_0} S_\Lambda (\lambda^{-1} |I|) |\Lambda|,$$

a Wegner estimate that gets better as the disorder increases.
Optimal Wegner estimate at the bottom of the spectrum at high disorder

Theorem

Let $H_{\omega,\lambda}$ be a crooked Anderson Hamiltonian with disorder $\lambda > 0$. 

Abel Klein  
Unique continuation principle for spectral projections
Optimal Wegner estimate at the bottom of the spectrum at high disorder

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Let $H_{\omega, \lambda}$ be a crooked Anderson Hamiltonian with disorder $\lambda > 0$. Then

$$E(\infty) := \lim_{t \to \infty} E(t) = \sup_{t \geq 0} E(t) > 0,$$

where $E(t) := \inf_{\sigma(H_0 + tu - W)}$. 

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$$E(\infty) := \lim_{t \to \infty} E(t) = \sup_{t \geq 0} E(t) > 0,$$

where $E(t) := \inf_{\sigma(H_0 + tu-W)} \sigma$. Moreover, for each $E_1 \in ]0, E(\infty)[$, there exists $\kappa = \kappa(E_1) > 0$, independent of $\lambda$, such that the following holds for all $\lambda > 0$:
Optimal Wegner estimate at the bottom of the spectrum at high disorder

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$$P^{(D)}_{\omega,\lambda,\Lambda}([-\infty, E_1]) U^{(\Lambda)} P^{(D)}_{\omega,\lambda,\Lambda}([-\infty, E_1]) \geq \kappa P^{(D)}_{\omega,\lambda,\Lambda}([-\infty, E_1]),$$
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and, for any interval $I \subset ] - \infty, E_1]$, \[E \left\{ \text{tr} P^{(D)}_{\omega, \lambda, \Lambda}(I) \right\} \leq C_{d, \delta_+, V_0^{(0)}} \left( \kappa^{-2}(1 + E_1) \right)^{2 + \frac{\log d}{\log 2}} S_{\Lambda}(\lambda^{-1} |I|) |\Lambda| \]
Lemma

Let $H_0$, $u_-$, $W$ be as in a crooked Anderson Hamiltonian, set $H(t) = H_0 + tu_-W$ for $t \geq 0$, and let $E(t) = \inf \sigma(H(t))$, $E(\infty) = \lim_{t \to \infty} E(t) = \sup_{t \geq 0} E(t)$.
A lower bound on $E(\infty)$

**Lemma**

Let $H_0, u_-, W$ be as in a crooked Anderson Hamiltonian, set $H(t) = H_0 + tu_- W$ for $t \geq 0$, and let $E(t) = \inf \sigma(H(t))$, $E(\infty) = \lim_{t \to \infty} E(t) = \sup_{t \geq 0} E(t)$. Then

$$E(t) \geq tu_- \delta_- M_d \left( 1 + \left( 2 V_0(0) + 2tu_- \right)^{\frac{2}{3}} \right)$$

for all $t \geq 0$, so we conclude that $E(\infty) \geq \sup_{t \in [0, \infty]} E(t)$. This lemma is proven from the Corollary to the QUCP.
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so we conclude that

$$E(\infty) \geq \sup_{t \in [0, \infty]} t\delta_- M_d \left(1 + \left(2V_\infty^{(0)} + 2tu_-\right)^{\frac{2}{3}} \right) > 0.$$
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The Theorem now follows using an extension of an abstract UCPS due to Boutet de Monvel, Lenz, and Stollmann (2011).
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Suppose $E(\infty) > E(0)$. Given $E_1 \in ]E(0), E(\infty)[$, let

$$\kappa = \kappa(H_0, Y, E_1) = \sup_{s > 0; \ E(s) > E_1} \frac{E(s) - E_1}{s} > 0.$$
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Then for all bounded operators $V \geq 0$ on $\mathcal{H}$ and Borel sets $B \subset ]-\infty, E_1]$ we have

$$\chi_B(H_0 + V) Y \chi_B(H_0 + V) \geq \kappa \chi_B(H_0 + V).$$
Proof of the abstract UCPSP

Fix $E_1 \in ]E(0), E(\infty)[$. For all Borel sets $B \subset ]-\infty, E_1]$ we have, writing $P_V(B) = \chi_B(H_0 + V)$,

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Since $E_1 \in ]E(0), E(\infty)[$, there is $s > 0$ such that $E(s) > E_1$. Then,

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and hence, using $V \geq 0$,

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We conclude that

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Localization in a fixed interval at high disorder

Theorem

Let $H_{\omega,\lambda}$ be an ergodic Anderson Hamiltonian with disorder $\lambda > 0$, and suppose the single-site probability distribution $\mu$ has a bounded density (or is uniformly Hölder continuous).

By complete localization on an interval $I$ we mean that for all $E \in I$ there exists $\delta(E) > 0$ such that we can perform the bootstrap multiscale analysis on the interval $(E - \delta(E), E + \delta(E))$, obtaining Anderson and dynamical localization.

This theorem was previously known only with a covering condition $U(\Lambda) \geq \alpha \chi_{\Lambda}$, $\alpha > 0$, in which case $E(\infty) = \infty$.

This theorem holds for crooked Anderson Hamiltonians with appropriate hypotheses on the single site probability distributions $\mu_j$. 
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Then, given $E_1 \in ]0, E(\infty)[$, there exists $\lambda(E_1) < \infty$, such that $H_{\omega,\lambda}$ exhibits complete localization on the interval $[0, E_1[$ for all $\lambda \geq \lambda(E_1)$. 

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References

doi:10.1007/s00220-013-1795-x