Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for random Schrödinger operators

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$$\Lambda_L(x) := \left\{ y \in \mathbb{R}^d ; |y - x|_\infty < \tfrac{L}{2} \right\}, \quad \text{with} \quad |x|_\infty := \max_{j=1,2,\ldots,d} |x_j| .$$

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- H_Λ denotes the restriction of H to the the box $\Lambda \subset \mathbb{R}^d$:

$$H_\Lambda = -\Delta_\Lambda + V_\Lambda \quad \text{on} \quad \mathrm{L}^2(\Lambda).$$

  - Δ_Λ is the Laplacian on Λ with either Dirichlet or periodic boundary condition.
  - V_Λ is the restriction of V to Λ..

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A UCPSP on a box Λ is an estimate of the form

$$\chi_I(H_\Lambda) W_\Lambda \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda) \quad \text{on} \quad \mathrm{L}^2(\Lambda),$$

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- Combes, Hislop and Klopp (2003): The UCPSP holds for bounded $\mathbb{Z}^d$-periodic potentials V and W, $W \geq 0$ with $W > 0$ on an open set, boxes $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with $L \in \mathbb{N}$, H_Λ with periodic boundary condition, with a constant $\kappa > 0$ depending on $\sup I$ (and d, V, W), but not on the box Λ. Their proof uses the unique continuation principle and Floquet theory.

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- Germinet and Klein (2013) proved a modified version of the CHK UCPSP, using Bourgain and Kenig's quantitative unique continuation principle and (some) Floquet theory, obtaining control of the constant κ in terms of the relevant parameters.

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- Rojas-Molina and Veselić (2013) proved, under the hypotheses of the Theorem, that for boxes $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, if ψ is an eigenfunction of H_Λ with eigenvalue $E \in]-\infty, E_0]$, then

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- Our Theorem is derived from the quantitative unique continuation principle as in Bourgain and Klein using the "dominant boxes" introduced by Rojas-Molina and Veselić.
- The UCPSP is a crucial ingredient for proving Wegner estimates for one and multi-particle Anderson Hamiltonians. The UCPSP replaces the covering condition.

Quantitative unique cont. principle (Bourgain-Klein)

Let $\Omega \subset \mathbb{R}^d$ open. Let $\psi \in \mathrm{H}^2(\Omega)$ and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by

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where V is a bounded real measurable function on Ω, $\|V\|_\infty \le K < \infty$.

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we have

$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\chi_\Omega\|_2}{\|\psi\chi_\Theta\|_2}\right)} \|\psi\chi_\Theta\|_2^2 \le \|\psi\chi_{B(x_0,\delta)}\|_2^2 + \|\zeta\chi_\Omega\|_2^2,$$

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$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \|\psi\|_2^2 \leq \sum_{k\in\mathbb{Z}^d, \Lambda_1(k)\subset\Lambda} \left\|\psi\chi_{B(y_k,\delta)}\right\|_2^2 + \|H_\Lambda\psi\|_2^2$$
$$= \left\|W^{(\Lambda)}\psi\right\|_2^2 + \|H_\Lambda\psi\|_2^2 .$$

Proof of the UCPSP

Let $E_0 > 0$ and $I = [E - \beta, E + \beta] \subset]-\infty, E_0]$ a closed interval. Since $H_\Lambda \geq -\|V\|_\infty$, we assume $E \in [-\|V\|_\infty, E_0]$ without loss of generality, so

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Let Λ be a box as in the Corollary and $\psi = \chi_I(H_\Lambda)\psi$ real-valued.

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Let Λ be a box as in the Corollary and $\psi = \chi_I(H_\Lambda)\psi$ real-valued. It follows from the Corollary applied to $H - E$ that

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}\|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \|(H_\Lambda - E)\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \beta^2\|\psi\|_2^2.$$

Proof of the UCPSP

Let $E_0 > 0$ and $I = [E - \beta, E + \beta] \subset]-\infty, E_0]$ a closed interval.
Since $H_\Lambda \geq -\|V\|_\infty$, we assume $E \in [-\|V\|_\infty, E_0]$ without loss of generality, so

$$\|V - E\|_\infty \leq \|V\|_\infty + \max\{E_0, \|V\|_\infty\} \leq K = 2\|V\|_\infty + E_0.$$

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If $\beta^2 \leq \gamma^2 := \frac{1}{2}\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}$, i.e., $|I| \leq 2\gamma$, we get

$$\gamma^2\|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2, \quad \text{i.e.,} \quad \gamma^2\chi_I(H_\Lambda) \leq \chi_I(H_\Lambda)W^{(\Lambda)}\chi_I(H_\Lambda).$$

Proof of the Corollary from the QUCP

Take $\Lambda = \Lambda_L(0)$ with $L \in \mathbb{N}_{\mathrm{odd}}$. We extend functions φ on Λ to functions $\widehat{V}$ and $\widetilde{\varphi}$ on $\mathbb{R}^d$ and V to a potential $\widehat{V}$ on $\mathbb{R}^d$ so

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Take $Y \in \mathbb{N}_{\text{odd}}$, $9 \leq Y < \frac{L}{2}$. Since L is odd, we have $\overline{\Lambda} = \bigcup_{k \in \Lambda \cap \mathbb{Z}^d} \overline{\Lambda_1(k)}$. It follows that for all $\varphi \in \mathrm{L}^2(\Lambda)$ we have

$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \widetilde{\varphi}_{\Lambda_Y(k)} \right\|_2^2 \leq (2Y)^d \left\| \varphi_\Lambda \right\|_2^2 .$$

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and note that, letting $\widehat{D} \subset \Lambda \cap \mathbb{Z}^d$ denote the collection of dominating sites,

$$\sum_{k \in \widehat{D}} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \geq \frac{1}{2} \left\| \psi_\Lambda \right\|_2^2 .$$

Proof of the Corollary-continued

If $k \in \widehat{D}$ we apply the QUCP with $\Omega = \Lambda_Y(k)$ and $\Theta = \Lambda_1(k)$, obtaining

$$\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \leq \left\| \psi_{B(y_{J(k)},\delta)} \right\|_2^2 + \left\| \widetilde{\zeta}_{\Lambda_Y(k)} \right\|_2^2,$$

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where $\zeta = (-\Delta + V)\psi$, Y is appropriately chosen, $Y \leq 40\sqrt{d} < \frac{L}{2}$, and the map $J\colon \widehat{D} \to \Lambda \cap \mathbb{Z}^d$ is defined appropriately so

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which implies (with a different constant $M_d > 0$)

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \left\|\psi_\Lambda\right\|_2^2 \leq \sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\|\psi\chi_{B(y_k,\delta)}\right\|_2^2 + \left\|\zeta_\Lambda\right\|_2^2.$$

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Remark: If $V^{(0)}$ is $q\mathbb{Z}^d$-periodic with $q \in \mathbb{N}$, and $y_j = j$, $v_j = v_0$, $\mu_j = \mu_0$ for all $j \in \mathbb{Z}^d$, then H_ω is the ergodic (usual) Anderson Hamiltonian.

Finite volume crooked Anderson Hamiltonians

We define finite volume crooked Anderson Hamiltonians on a box $\Lambda = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ and $L > 0$, with either Dirichlet or periodic boundary condition, by

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We also set

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$$W(x) := \sum_{j \in \mathbb{Z}^d} \chi_{B(y_j,\delta_-)}(x) \quad \text{and} \quad W^{(\Lambda)}(x) := \sum_{j \in \mathbb{Z}^d, \Lambda_1(j) \subset \Lambda} \chi_{B(y_j,\delta)}.$$

Remark and notation

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An optimal Wegner estimate for Anderson Hamiltonians is an estimate of the form

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$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\Lambda}(I)\right\} \le C_{d,\delta_+,\|V^{(0)}\|_\infty}\left(u_-^{-2}\gamma^{-4}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|.$$

UCPSP $\Longrightarrow$ Optimal Wegner estimate

The theorem (optimal Wegner estimates) follows from the UCPSP theorem and the following lemma.

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Then

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\Lambda}(I)\right\} \leq C_{d,\delta_+,\|V^{(0)}\|_\infty} \left(\kappa^{-2}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|\,.$$

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We may now adapt an argument in in Combes, Hislop, Klopp obtaining

$$\mathbb{E} \operatorname{tr} P \le C_{d,\delta_+,V_\infty^{(0)}} \left(\kappa^{-2}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|\,.$$

Multi-particle Anderson Hamiltonians

The n-particle Anderson Hamiltonian is the random Schrödinger operator

$$H_\omega^{(n)} := H_{0,\omega}^{(n)} + U \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^{nd}), \quad \text{where} \quad H_{0,\omega}^{(n)} := -\Delta^{(n)} + V_\omega^{(n)}.$$

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5. Given a one-particle box Λ, we will use $\mathbb{E}_\Lambda$ and $\mathbb{P}_\Lambda$ to denote the expectation and probability with respect to the probability distribution of the random variables $\{\omega_k\}_{k \in \widehat{\Lambda}}$.

Finite volume multi-particle Anderson Hamiltonians

Given an n-particle box $\mathbf{\Lambda} = \mathbf{\Lambda}_L^{(n)}(\mathbf{a})$, we define the corresponding finite volume Anderson Hamiltonian $H_{\omega,\mathbf{\Lambda}}^{(n)}$ on $\mathrm{L}^2(\mathbf{\Lambda})$ by

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We set

$$R_{\omega,\mathbf{\Lambda}}^{(n)}(z) = (H_{\omega,\mathbf{\Lambda}}^{(n)} - z)^{-1} \quad \text{for} \quad z \notin \sigma\left(H_{\omega,\mathbf{\Lambda}}^{(n)}\right).$$

Wegner estimate for multi-particle Anderson Hamiltonians

Theorem

Let $n \in \mathbb{N}$ and $E_+ > 0$. There exist constants $\gamma_{n,E_+} > 0$ and C_{n,E_+}, such that, for all n-particle boxes $\mathbf{\Lambda} = \mathbf{\Lambda}_L^{(n)}(\mathbf{a})$ with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^{nd}$ and $L \geq 114\sqrt{nd}$ and all intervals $I \subseteq [0, E_+)$ with $|I| \leq 2\gamma_{n,E_+}$, we have

$$\mathbb{E}_{\Lambda_L(a_i)} \left\{ \operatorname{tr} \chi_I \left(H_{\omega,\mathbf{\Lambda}}^{(n)} \right) \right\} \leq C_{n,E_+} \|\rho\|_\infty |I| L^{nd} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

In particular, for any $E \leq E_+$, $0 < \varepsilon \leq \gamma_{n,E_+}$, and $i = 1, 2, \ldots, n$, we have

$$\mathbb{P}_{\Lambda_L(a_i)} \left\{ d\left(\sigma(H_{\omega,\mathbf{\Lambda}}^{(n)}), E\right) \leq \varepsilon \right\} \leq 2 C_{n,E_+} \|\rho\|_\infty \varepsilon L^{nd}.$$

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Hislop and Klopp: similar Wegner estimate taking expectation over all random variables.

Proof of multi-particle Wegner estimate

Let $\mathbf{\Lambda} = \mathbf{\Lambda}_L^{(n)}(\mathbf{a})$, $\Lambda_i = \Lambda_L(a_i)$.

$$V_{\omega,\mathbf{\Lambda}}^{(n)}(\mathbf{x}) = \sum_{i=1}^{n} V_{\omega,\Lambda_i}^{(1)}(x_i) = \sum_{i=1}^{n} \sum_{k \in \widehat{\Lambda}_i} \omega_k u(x_i - k) = \sum_{k \in \mathbb{Z}^d} \omega_k \theta_k^{(\mathbf{\Lambda})}(\mathbf{x}),$$

$$\theta_k^{(\mathbf{\Lambda})}(\mathbf{x}) = \sum_{\{i;\, k \in \widehat{\Lambda}_i\}} u(x_i - k) \geq u_- \sum_{\{i;\, k \in \widehat{\Lambda}_i\}} \chi_{\Lambda_{\delta_-}^{(1)}(k)}(x_i).$$

Fix $q \in \{1, 2, \ldots, n\}$, we have

$$H_{\omega,\mathbf{\Lambda}}^{(n)} = -\Delta_{\mathbf{\Lambda}}^{(n)} + U_{\mathbf{\Lambda}} + \sum_{k \in \mathbb{Z}^d \setminus \widehat{\Lambda_q}} \omega_k \theta_k^{(\mathbf{\Lambda})} + \sum_{k \in \widehat{\Lambda_q}} \omega_k \theta_k^{(\mathbf{\Lambda})}.$$

Then for $\mathbf{x} \in \mathbf{\Lambda}$ we have (with $\eta = \min\{\frac{\delta_-}{2}, \frac{1}{2}\}$)

$$W^{(\mathbf{\Lambda})}(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbf{\Lambda} \cap \mathbb{Z}^{nd}} \chi_{B^{(n)}(\mathbf{k},\eta)}(\mathbf{x}) \leq u_-^{-1} \sum_{k \in \widehat{\Lambda_q}} \theta_k^{(\mathbf{\Lambda})}(\mathbf{x})$$

Proof of multi-particle Wegner estimate-cont.

Fix $E_+ > 0$. It follows from the UCPSP Theorem that for any interval $I \subseteq [0, E_+)$ with $|I| \leq 2\gamma_{n,E_+}$ we have

$$\begin{aligned} \chi_I(H^{(n)}_{\omega,\boldsymbol{\Lambda}}) &\leq \gamma^{-2}_{n,E_+} \chi_I(H^{(n)}_{\omega,\boldsymbol{\Lambda}}) W^{(\Lambda)} \chi_I(H^{(n)}_{\omega,\boldsymbol{\Lambda}}) \\ &\leq u_-^{-1} \gamma^{-2}_{n,E_+} \chi_I(H^{(n)}_{\omega,\boldsymbol{\Lambda}}) \left(\sum_{k \in \widehat{\Lambda_q}} \theta_k^{(\boldsymbol{\Lambda})} \right) \chi_I(H^{(n)}_{\omega,\boldsymbol{\Lambda}}), \end{aligned}$$

where $\quad \gamma^2_{n,E_+} = \frac{1}{2} \eta^{M_{nd}\left(1+K^{\frac{2}{3}}\right)} \quad$ with $\quad K = n(n-1)\|\widetilde{U}\|_\infty + 2M_+\delta_+^d + E_+.$

The Wegner estimate can now be proved following as in one-particle case, averaging only the random variables $\{\omega_i\}_{i \in \widehat{\Lambda_q}}$.

Wegner estimates at high disorder

Let $H_{\omega,\lambda} = H_0 + \lambda V_\omega$ be a crooked Anderson Hamiltonian, where $\lambda > 0$ is the disorder parameter.

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We can make explicit the dependence on λ in the Wegner estimate:

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\lambda,\Lambda}(I)\right\} \le C_{E_0} \mathrm{e}^{c_{E_0}\left(1+\lambda^{\frac{2}{3}}\right)} S_\Lambda(\lambda^{-1}|I|)\,|\Lambda|\,.$$

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If we use the UCPSP for H_0, as in Combes, Hislop and Klopp, we get

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and, for any interval $I \subset]-\infty, E_1]$,

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A lower bound on $E(\infty)$

Lemma

Let H_0, u_-, W *be as in a crooked Anderson Hamiltonian, set* $H(t) = H_0 + tu_-W$ *for* $t \geq 0$, *and let* $E(t) = \inf \sigma(H(t))$, $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t)$.

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This lemma is proven from the Corollary to the QUCP.

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Then for all bounded operators $V \geq 0$ on $\mathscr{H}$ and Borel sets $B \subset]-\infty, E_1]$ we have

$$\chi_B(H_0 + V)\, Y\, \chi_B(H_0 + V) \geq \kappa \chi_B(H_0 + V).$$

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Fix $E_1 \in]E(0), E(\infty)[$. For all Borel sets $B \subset]-\infty, E_1]$ we have, writing $P_V(B) = \chi_B(H_0 + V)$,

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This theorem holds for crooked Anderson Hamiltonians with appropriate hypotheses on the single site probability distributions μ_j.

References

A. Klein: Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators, Comm. Math Phys. **323**, 1229-1246 (2013). doi:10.1007/s00220-013-1795-x

A. Klein and S. T. Nguyen: Bootstrap multiscale analysis and localization for multi-particle continuous Anderson Hamiltonians, J. Spectr. Theory, to appear. arXiv:1311.4220