Full counting statistics of return to equilibrium

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Physical picture

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We consider the context of return to equilibrium

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0th law of thermodynamics:

an isolated systems out of equilibrium reaches "rapidly enough" an equilibrium state (characterized by macroscopic parameters)

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two interacting isolated systems out of equilibrium reach "rapidly enough" an equilibrium state (characterized by macroscopic parameters)

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0th law of thermodynamics:

two interacting isolated systems out of equilibrium reach "rapidly enough" an equilibrium state (characterized by macroscopic parameters)

1st law - conservation of energy:

$$\triangle Q_1 = \triangle Q_2$$

Slightly diffrent situation: system 1= small system $\mathcal{S}$

system 2 = reservoir $\mathcal{R}$

Statistical mechanics: derive macroscopic law from (quantum) microscopic law

Mathematical setting

small system $\mathcal{S}$

$\mathcal{H}_{\mathcal{S}}$ Hilbert space $dim\mathcal{H}_{\mathcal{S}} < \infty$

$H : \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{S}}$

$\mathcal{O}_{\mathcal{S}} = \mathcal{B}(\mathcal{H}_{\mathcal{S}})$

$\omega_{\mathcal{S}} : \mathcal{O}_{\mathcal{S}} \to \mathbb{C} \quad \omega_{\mathcal{S}}(A) = \mathrm{tr}(\rho_{\mathcal{S}} A)$

$$\begin{array}{rccc} \tau_{\mathcal{S}}^t : & \mathcal{O}_{\mathcal{S}} \to & \mathcal{O}_{\mathcal{S}} \\ & A \to & \tau_t(A) = A_t := e^{\mathrm{i}tH_{\mathcal{S}}} A e^{-\mathrm{i}tH_{\mathcal{S}}} \end{array}$$

$\tau_{\mathcal{S}}^t$ strongly continous in t with generator $\delta_{\mathcal{S}} = \mathrm{i}[H_{\mathcal{S}}, -]$

$(\mathcal{O}_{\mathcal{S}}, \tau_{\mathcal{S}}^t, \omega_{\mathcal{S}})$ is a dynamical system

equilibrium state: $\omega_\beta(A) := \frac{tr(\rho_\beta A)}{\mathrm{tr}\rho_\beta} \quad \rho_\beta := e^{-\beta H_{\mathcal{S}}}$

Mathematical setting

reservoir $\mathcal{R}$

$\mathcal{H}_\mathcal{R}$ Hilbert space,
$H : \mathcal{H}_\mathcal{R} \to \mathcal{H}_\mathcal{R}$

$\mathcal{O}_\mathcal{R} \subset \mathcal{B}(H_\mathcal{R})$
$\omega_\mathcal{R} : \mathcal{O}_\mathcal{R} \to \mathbb{C} \quad \omega_\mathcal{R}(A) = \mathrm{tr}(\rho_R A)$

$$\begin{aligned} \tau^t_\mathcal{R} : \quad \mathcal{O}_\mathcal{R} &\to \quad \mathcal{O}_\mathcal{R} \\ A &\to \quad \tau_t(A) = A_t := e^{\mathrm{i}tH_\mathcal{R}} A e^{-\mathrm{i}tH_\mathcal{R}} \end{aligned}$$

$\tau^t_\mathcal{R}$ strongly continous in t with generator $\delta_\mathcal{R} = \mathrm{i}[H_\mathcal{R}, -]$

Mathematical setting

reservoir $\mathcal{R}$

$(\mathcal{O}_{\mathcal{R}}, \tau_{\mathcal{R}}^{t}, \omega_{\mathcal{R}})$
$\mathcal{O}_{\mathcal{R}}$ - $C^{*}-$ algebra
$\tau_{\mathcal{R}}^{t} : \mathcal{O}_{\mathcal{R}} \to \mathcal{O}_{\mathcal{R}}$ $*$- automorphism strongly continuous in t with generator $\delta_{R} : D(\delta_{R}) \subset \mathcal{O}_{\mathcal{R}} \to \mathcal{O}_{\mathcal{R}}$
$\omega_{\mathcal{R},\beta}$ equilibrium state: $(\tau_{\mathcal{R}}^{t}, \beta)$ KMS State (hence faithful)
($\omega_{\mathcal{R},\beta} := \frac{tr(\rho_{\beta} A)}{\mathrm{tr} \rho_{\beta}}$ $\quad \rho_{\beta} := e^{-\beta H_{\mathcal{R}}}$ when the above expression is well defined)

Mathematical setting

Full system free dynamics:

$(\mathcal{O}, \tau_0, \omega_0)$

with $\mathcal{O} := \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{R}}$

$\tau_0 := \tau_{\mathcal{S}} \otimes \tau_{\mathcal{R}}$

$\omega_0 := \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R},\beta}$

$\omega_{\beta,0} := \omega_{\mathcal{S},\beta} \otimes \omega_{\mathcal{R},\beta}$

Full system interacting dynamics: $(\mathcal{O}, \tau_\lambda, \omega_0)$

τ_λ with generator $\delta = \delta_0 + i\lambda[V, -]$, $V = V^*$, $V \in \mathcal{O}$

To simplify notation $\omega_0 =: \omega$

Mathematical setting

1st law

$$\triangle Q_{\mathcal{S}}(\lambda, t) = \omega(\tau_\lambda^t(H_{\mathcal{S}})) - \omega(H_{\mathcal{S}}) = \int_0^t \omega(\tau_\lambda^s(\Phi_{\mathcal{S}}))\mathrm{d}s$$

where $\Phi_{\mathcal{S}} = -\delta_{\mathcal{S}}(\lambda V)$

Mathematical setting

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where $\Phi_{\mathcal{S}} = -\delta_{\mathcal{S}}(\lambda V)$

$$\triangle Q_{\mathcal{R}}(\lambda, t) = -\int_0^t \omega(\tau_\lambda^s(\Phi_{\mathcal{R}}))\mathrm{d}s$$

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Mathematical setting

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where $\Phi_{\mathcal{S}} = -\delta_{\mathcal{S}}(\lambda V)$

$$\triangle Q_{\mathcal{R}}(\lambda, t) = -\int_0^t \omega(\tau_\lambda^s(\Phi_{\mathcal{R}}))\mathrm{d}s \; (= -\omega(\tau_\lambda^t(H_{\mathcal{R}})) + \omega(H_{\mathcal{R}}))$$

where $\Phi_{\mathcal{R}} = -\delta_{\mathcal{R}}(\lambda V)$

Mathematical setting

1st law

$\triangle Q_{\mathcal{S}}(\lambda, t) = \triangle Q_{\mathcal{R}}(\lambda, t) + \lambda(\omega(\tau_\lambda^t(V)) - \omega(V)$

We want to take first $t \to \infty$ then $\lambda \to 0$

Proposition (well known, BR2)

If $V \in \mathcal{O}$, then there exists (τ_λ, β)-KMS state $\omega_{\beta,\lambda}$. Moreover

$$\lim_{\lambda \to 0} \omega_{\beta,\lambda} = \omega_{\beta,0}$$

Mathematical setting

1st law

Assumptions - $V \in \mathcal{O}$: (hypothesis of previous proposition)
- $(\mathcal{O}, \tau_\lambda, \omega_{\beta,\lambda})$ is mixing for λ small enough i.e.

$$\lim_{t\to\infty} \xi(\tau_\lambda^t(A)) = \omega_{\beta,\lambda}(A)$$

for all $\xi \in \mathcal{N}_{\omega_{\beta,\lambda}}$ (normal states)

Mathematical setting

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Remark Confined system are never mixing

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$$\triangle Q_{\mathcal{S}}(\lambda, t) = \triangle Q_{\mathcal{R}}(\lambda, t) + \lambda\left(\omega(\tau_\lambda^t(V)) - \omega(V)\right)$$

Mathematical setting

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$$\triangle Q_{\mathcal{S}}(\lambda) = \triangle Q_{\mathcal{R}}(\lambda) + \lambda\left(\omega_\lambda(V) - \omega(V)\right)$$

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Conclusion

If V is $\mathcal{O}$ and $(\mathcal{O}, \tau_\lambda, \omega_{\beta,\lambda})$ is mixing then

$$\lim_{\lambda\to 0}\lim_{t\to\infty} \triangle Q_{\mathcal{S}}(\lambda, t) = \lim_{\lambda\to 0}\lim_{t\to\infty} \triangle Q_{\mathcal{R}}(\lambda, t)$$

Theorem (Jakšić, A.P., J. Panangaden, C-A. Pillet '14)

Let $(\mathcal{O}, \tau_\lambda, \omega)$ *as before* ($\tau_\lambda = \tau_0 + \mathrm{i}[V, -]$).
Assume:
- $V \in \mathcal{O}$,
- $t \to \tau_\lambda^t(V)$ *extends to an entire analytic function,*
- $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ *is mixing for* $0 < |\lambda| < \lambda_0$. *Then*

$$\mathbb{P}_{\mathcal{S}} := \lim_{\lambda \to 0} \lim_{t \to \infty} \mathbb{P}_{\mathcal{S},\lambda,t} = \lim_{\lambda \to 0} \lim_{t \to \infty} \mathbb{P}_{\mathcal{R},\lambda,t} =: \mathbb{P}_{\mathcal{R}}$$

Full counting statistics

Full Counting Statistic-
several results in the context non-equilibrium /transport phenomena/fluctuation relations:

[Lesovik, Levitov 93][Levitov, Lee,Lesovik 96]

[Kurchan 00] [Klich 03][deRoeck, Maes 04] [Derezinski, de Roeck, Maes 07], [Avron Bachmann Graf Klich 07] [Tasaki Matsui 03] and others

Full counting statistics

Small system $\mathcal{S}$: $H_{\mathcal{S}} = \sum_j e_j P_{e_j}$ where $e_j \in \sigma(H_{\mathcal{S}})$ P_{e_j} associated spectral projections

At time 0 we measure energy with outcome e_j with probability $\omega(P_{e_j})$

Then the reduced state is

$$\omega_{am} = \frac{1}{\omega(P_{e_j})} P_{e_j} \rho_{\mathcal{S}} P_{e_j} \otimes \omega_{\mathcal{R}}$$

Let evolve for time t, and measure again. The outcome will be e_k with probability

$$\omega_{am}(\tau_\lambda^t(P_{e_k})) = \frac{1}{\omega(P_{e_j})} \left(P_{e_j} \rho_{\mathcal{S}} P_{e_j} \otimes \omega_{\mathcal{R}} \right) \left(\tau_\lambda^t(P_{e_k}) \right)$$

Full counting statistics

hence the joint probability of measuring e_j, e_k is

$$P_{e_j}\rho_{\mathcal{S}}P_{e_j}\otimes\omega_{\mathcal{R}}(\tau_\lambda^t(P_{e_k}))$$

Full Counting statistic of energy transfer is the atomic probability measure on $\mathbb{R}$ defined by

$$\mathbb{P}_{\mathcal{S},\lambda,t}(\phi)=\sum_{e_j-e_k=\phi}P_{e_j}\rho_{\mathcal{S}}P_{e_j}\otimes\omega_{\mathcal{R}}(\tau_\lambda^t(P_{e_k}))$$

(probability distribution of the energy change measured with the protocol above)

$$\triangle Q_{\mathcal{S}}(\lambda,t)=\int\phi\,\mathbb{P}_{\mathcal{S},\lambda,t}(\phi)$$

Full counting statistics

Under mixing assumption

$$\mathbb{P}_{\mathcal{S},\lambda} := \lim_{t\to\infty} \mathbb{P}_{\mathcal{S},\lambda,t} = \sum_{e_j - e_k = \phi} \mathrm{tr}(\rho_{\mathcal{S}} P_{e_j})\omega_\lambda(P_{e_k})$$

$$\mathbb{P}_{\mathcal{S}} := \lim_{\lambda\to 0} \mathbb{P}_{\mathcal{S},\lambda} = \sum_{e_j - e_k = \phi} \mathrm{tr}(\rho_{\mathcal{S}} P_{e_j})\mathrm{tr}(\rho_{\mathcal{S}} P_{e_k})$$

Full counting statistics

Reservoir $\mathcal{R}$: Let's pretend $\mathcal{R}$ is a finite system. Let's give a parallel description to the one of system $\mathcal{S}$

$H_{\mathcal{R}} = \sum_k \epsilon_k P_{\epsilon_k}$

$$\mathbb{P}_{\mathcal{R},\lambda,t}(\phi) = \sum_{\epsilon_j - \epsilon_k = \phi} \mathrm{tr}(e^{-\mathrm{i}tH_\lambda}(\rho_{\mathcal{S}} \otimes \rho_{\mathcal{R}} P_{\epsilon_k})e^{\mathrm{i}tH_\lambda} \mathbb{1} \otimes P_{\epsilon_k})$$

$$\int e^{\mathrm{i}\alpha\phi} \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\phi) = \sum_{k,j} e^{i\alpha(\epsilon_j - \epsilon_k)} \mathrm{tr}(\mathbb{1} \otimes P_{\epsilon_k} e^{-\mathrm{i}tH_\lambda}(\mathbb{1} \otimes P_{\epsilon_j})(\rho_{\mathcal{S}} \otimes \rho_{\mathcal{R}})e^{\mathrm{i}tH_\lambda})$$

$$= \mathrm{tr}((\mathbb{1} \otimes \rho_{\mathcal{R}}^{i\frac{\alpha}{\beta}})(e^{-itH_\lambda} \rho_{\mathcal{S}} \otimes \rho_{\mathcal{R}}^{1-\mathrm{i}\frac{\alpha}{\beta}} e^{\mathrm{i}tH_\lambda}))$$

$$= \omega(\triangle_{\eta_t|\eta}^{\mathrm{i}\frac{\alpha}{\beta}}(\mathbb{1})) \quad \eta := \mathbb{1} \otimes \rho_{\mathcal{R}}$$

Full counting statistics- relative modular operator

Classical setting: Radon-Nikodym derivative
$\nu << \mu$ one can define $\frac{d\mu}{d\nu}$ with property

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$$\int fg\mathrm{d}\mu = \int f\frac{\mathrm{d}\mu}{\mathrm{d}\nu}g\mathrm{d}\nu$$

Full counting statistics- relative modular operator

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$\nu << \mu$ one can define $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ with property

$$\mu(fg) = \nu(f\frac{\mathrm{d}\mu}{\mathrm{d}\nu}g)$$

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Quantum setting:
Given two states ν, μ, denote by ρ_ν, ρ_μ the associated density matrices . Define

$$\triangle_{\mu|\nu}(A) := \rho_\nu A \rho_\mu^{-1}$$

then

$$\mu(AB) = \nu(A \triangle_{\mu|\nu}(B)) \quad \textit{for all } A, B \in \mathcal{O}$$

Full counting statistics- relative modular operator

One easly shows :

$$\omega(\triangle_{\eta_t|\eta}^{\mathrm{i}\frac{\alpha}{\beta}}(\mathbb{1})) = \mathrm{tr}((\mathbb{1} \otimes \rho_{\mathcal{R}}^{i\frac{\alpha}{\beta}})(e^{-i\lambda tH_\lambda}\rho_{\mathcal{S}} \otimes \rho_{\mathcal{R}}^{1-\mathrm{i}\frac{\alpha}{\beta}} e^{\mathrm{i}tH_\lambda}))$$

Remark In the canonical GNS representation associated to
$\omega = \mathrm{tr}(\rho_\omega -)$ faithful
$\mathcal{O} = \mathcal{H}_{\mathcal{O}}$, $(A, B)_\omega = \omega(A^* B)$

$\triangle_{\mu|\nu} : \mathcal{H}_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$

$\triangle_{\mu|\nu}(\psi_A) = \triangle_{\mu|\nu}(A) = \rho_\nu A \rho_\mu^{-1}$ is a self adjoint operator.

Full counting statistics- relative modular operator

By algebraic theory, $\triangle_{\eta_t|\eta}$ can be defined in a general setting (infinitely extended reservoir) and it is by construction a selfajoint operator on $\mathcal{H}_{\mathcal{O}}$

We take as definition of $\mathbb{P}_{\mathcal{R},\lambda,t}$ to be

$$\int e^{\mathrm{i}\alpha\phi}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t} := \omega(\triangle_{\eta_t|\eta}^{\mathrm{i}\frac{\alpha}{\beta}}(\mathbb{1})) \quad \eta := \mathbb{1}\otimes\omega_{\mathcal{R}}$$

In other words:

$$\mathbb{P}_{\mathcal{R},\lambda,t} \textit{ is the spectral measure of } -\frac{1}{\beta}\log\triangle_{\eta_t|\eta} \quad \eta := \mathbb{1}\otimes\omega_{\mathcal{R}}$$

Theorem (Jakšić, A.P., J. Panangaden, C-A. Pillet '14)

Let $(\mathcal{O}, \tau_\lambda, \omega)$ *as before* ($\tau_\lambda = \tau_0 + \mathrm{i}[V, -]$).
Assume:
- $V \in \mathcal{O}$,
- $t \to \tau_\lambda(V)$ *extends to an entire analytic function,*
- $(\mathcal{O}, \tau_\lambda^t, \omega_\lambda)$ *is mixing for* $0 < |\lambda| < \lambda_0$. *Then*

$$\mathbb{P}_{\mathcal{S}} := \lim_{\lambda\to 0}\lim_{t\to\infty} \mathbb{P}_{\mathcal{S},\lambda,t} = \lim_{\lambda\to 0}\lim_{t\to\infty} \mathbb{P}_{\mathcal{R},\lambda,t} =: \mathbb{P}_{\mathcal{R}}$$

a word about the proof

-write

$$\omega(\triangle_{\eta_t|\eta}^{\mathrm{i}\frac{\alpha}{\beta}}(\mathbb{1})) = \left(\hat{\Omega}_{\frac{\alpha}{\beta}}, e^{-\mathrm{i}tL_\lambda}\Omega_{\eta,\frac{\alpha}{\beta}}\right)$$

- the above identity make sense for $\frac{\alpha}{\beta} = \frac{1}{2} + \mathrm{i}s$, $s \in \mathbb{R}$, extend the identity by analyticity

- use established result

$(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ mixing iff

$$w - \lim_{t\to\infty} e^{-\mathrm{i}tL_\lambda} = \frac{1}{||\Omega_\lambda||}|\Omega_\lambda\rangle\langle\Omega_\lambda|$$

Remarks

-Mixing hypothesis has been proved for many physical models for both bosonic and fermionc resevoirs
bosonic reservoir [BachFröhlich SigalS 00], [Derezinski Jakšić 03] [FröhlichMerkli04] [deRoeckKupianen11], fermionic reservoirs [AizenstadtMalyshev87], [Aschbacher,Jakšić PautratPillet07], [FröhlichMerkliUeltschi03] [FröhlichMerkliSchwarzUeltschi03][Jakšić Pillet97]

(locally interacting fermionic system [BotvichMalyshev83], [Jakšić OgataPillet07])
- $V \in \mathcal{O}$ restricts our analysis to bounded perturbations- in concrete models V unbounded for bosonic reservoirs.