Landauer's Principle in Quantum Statistical Mechanics

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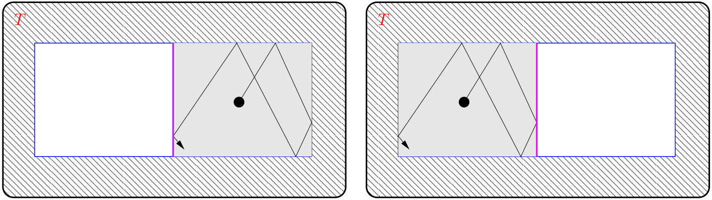
1. Introduction

Thermodynamic "derivation" of Landauer's Principle

The ideal gas 1-bit memory ($pV = k_B T$)



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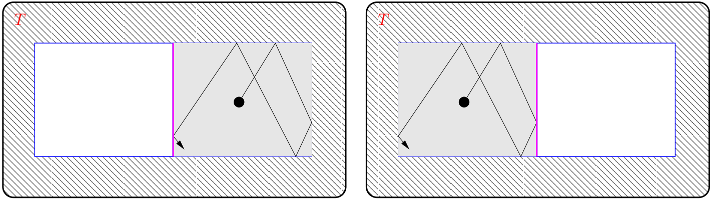

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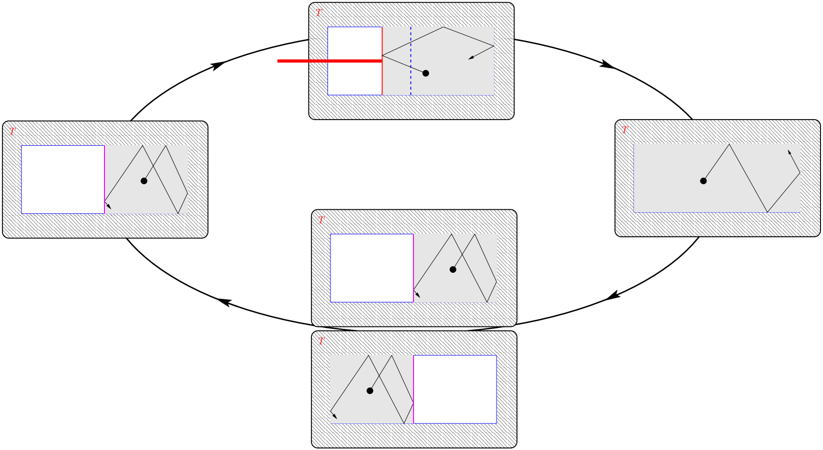
Assume the reset operation **(0 or 1)** $\to$ 0 can be performed with energy cost

$$\Delta E_{\mathrm{reset}}$$

by whatever process

Thermodynamic “derivation” of Landauer’s Principle

Build a cyclic process



Thermodynamic "derivation" of Landauer's Principle

Work extracted during isothermal quasi-static expansion

$$W = \int_{V/2}^{V} p\, dV = \int_{V/2}^{V} \frac{k_{\mathrm{B}} T}{V}\, \mathrm{d}V = k_{\mathrm{B}} T \log 2$$

The second law imposes

$$\Delta E_{\mathrm{reset}} \geq k_{\mathrm{B}} T \log 2 = \Delta E_{\mathrm{Landauer}}$$

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[Landauer ’61] The energy injected in the reset process is released as heat in the reservoir. Thus $\Delta E_{\mathrm{Landauer}}$ is the minimal energy dissipated by a reset operation. Moreover

$$\Delta E_{\mathrm{Landauer}} = T \Delta S$$

ΔS being the decrease in entropy of the system in the reseting process (erasing entropy).

2. Landauer's Principle in statistical mechanics

Landauer's Principle in statistical mechanics [Reeb-Wolf '14]

Finite quantum system $\mathcal{S}$ coupled to finite reservoir $\mathcal{R}$ at temperature $T > 0$

- Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}$ (finite dimensional)
- Reservoir Hamiltonian $H_\mathcal{R}$
- Initial state $\omega_i = \rho_i \otimes \nu_i$

$$\nu_i = \mathrm{e}^{-(\beta H_\mathcal{R} + \log Z)}, \qquad \beta = \frac{1}{k_\mathrm{B} T}, \qquad Z = \mathrm{tr}\left(\mathrm{e}^{-\beta H_\mathcal{R}}\right)$$

- Unitary state transformation U
- Final state $\omega_f = U\omega_i U^*$

$$\rho_f = \mathrm{tr}_{\mathcal{H}_\mathcal{R}}(\omega_f), \qquad \nu_f = \mathrm{tr}_{\mathcal{H}_\mathcal{S}}(\omega_f)$$

- Energy dissipated in the reservoir $\mathcal{R}$:

$$\Delta Q = \mathrm{tr}((\nu_f - \nu_i)H_\mathcal{R})$$

- Decrease in entropy of the system $\mathcal{S}$:

$$\Delta S = S(\rho_i) - S(\rho_f)$$

where $S(\rho) = -k_\mathrm{B}\,\mathrm{tr}\,(\rho \log \rho)$ is the von Neumann entropy of ρ

Landauer's Principle in statistical mechanics

Landauer's bound [Reeb-Wolf '14]

$$\Delta Q = T(\Delta S + \sigma), \qquad \sigma \geq 0 \tag{1}$$

$\sigma = 0$ iff $\Delta Q = T\Delta S = 0$, in which case one has

$$\nu_f = \nu_i$$

and ρ_f is unitarily equivalent to ρ_i.

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Remark 1. If $\mathcal{S}$ is a qubit,

$$\rho_i = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \rho_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then the transformation $\rho_i \to \rho_f$ implements the state change **(**0 **or** 1**)** $\to$ 0 and

$$T\Delta S = k_{\mathrm{B}} T \log 2$$

However, this transition can not be induced by a finite reservoir at positive temperature (more later).

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Remark 2. Von Neumann entropy is the quantum version of Shannon information theoretic entropy. It only coincides with thermodynamic (Clausius) entropy for thermal equilibrium states.

Landauer's Principle in statistical mechanics

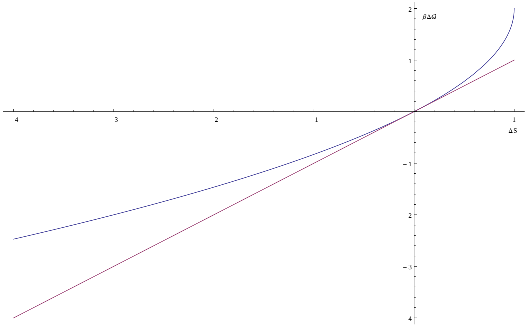
Remark 3. Landauer's bound is **not optimal** for finite dimensional reservoirs. Part of the analysis in [Reeb-Wolf '14] consists in refining it. A simple improvement, based on the well-known inequality

$$\sup_{A \neq 0} \frac{|\mathrm{tr}\,((\omega - \nu)A)|^2}{\|A\|^2} \leq 2\,\mathrm{tr}\,(\omega(\log\omega - \log\nu))$$

is given by

$$\beta\Delta Q \geq \left(1 + \frac{1 - \sqrt{1 - \Delta S/S_0}}{1 + \sqrt{1 - \Delta S/S_0}}\right)\Delta S$$

where $S_0 = \beta^2\ell^2/8$ and $\ell = \mathrm{diam\,spec}(H_{\mathcal{R}})$



Landauer's Principle in statistical mechanics

The last remark calls for considering infinite reservoirs which allow for $\ell = \infty$. Indeed, Reeb and Wolf made the following

Conjecture

Landauer's Principle can probably be formulated within the general statistical mechanical framework of C^* and W^* dynamical systems and an equality akin to (1) can possibly be proven. Note that in this framework the mutual information can be written as a relative entropy and the heat flow as a derivation w.r.t. the dynamical semigroup.

We shall see that this conjecture follows from well known results from the '70, in particular Araki's perturbation theory of KMS structure.

To simplify the notation, we

- set $k_{\mathrm{B}} = 1$;
- omit tensored identity, i.e., write A and B for $A \otimes I$ and $I \otimes B$;
- only consider C^*-dynamical systems. Extension to the W^*-setting is easy.

The algebraic framework

C^*-dynamical system $(\mathcal{O}, \tau)$

- Unital C^*-algebra $\mathcal{O}$ (observables).
- Strongly continuous group $t \mapsto \tau^t = \mathrm{e}^{t\delta} \in \mathrm{Aut}(\mathcal{O})$ (Heisenberg dynamics).

State ω

- Positive linear functional $\omega : \mathcal{O} \to \mathbb{C}$ such that $\omega(\mathbb{1}) = 1$.
- Schrödinger evolution $\omega_t = \omega \circ \tau^t$.
- τ-invariant if $\omega_t = \omega$ for all t.

GNS-Representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$

- $\mathcal{H}_\omega$ a Hilbert space.
- $\pi_\omega : \mathcal{O} \to \mathcal{B}(\mathcal{H}_\omega)$ a $*$-morphism.
- $\Omega_\omega \in \mathcal{H}_\omega$ a unit vector such that $\pi_\omega(\mathcal{O})\Omega_\omega$ is dense in $\mathcal{H}_\omega$.
- $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$.
- $\mathcal{N}_\omega$ set of ω-normal states $A \mapsto \mathrm{tr}(\rho\pi_\omega(A))$ (ρ a density matrix on $\mathcal{H}_\omega$).
- ω τ-invariant $\Rightarrow \pi_\omega \circ \tau^t(A) = \mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_\omega}$. The Liouvillean L_ω is self-adjoint and $L_\omega\Omega_\omega = 0$.

The algebraic framework

Ergodic/Mixing state

- ω is ergodic for $(\mathcal{O}, \tau)$ if, for all $\zeta \in \mathcal{N}_\omega$ and $A \in \mathcal{O}$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \zeta\circ\tau^s(A)\,\mathrm{d}s = \omega(A)$$

- and mixing if

$$\lim_{t\to\infty} \zeta\circ\tau^t(A) = \omega(A)$$

Thermal equilibrium state ω at inverse temperature $\beta = 1/T$

- (τ, β)-KMS state: $\omega(A\tau^{t+\mathrm{i}\beta}(B)) = \omega(\tau^t(B)A)$.
- τ-invariant.
- $\pi_\omega(A)\Omega_\omega = 0 \Rightarrow \pi_\omega(A) = 0$.

Relative entropy of positive linear functionals

- Finite dimensional case: $S(\zeta_1|\zeta_2) = \mathrm{tr}(\zeta_1(\log\zeta_1 - \log\zeta_2))$.
- Extends to general C^*/W^* setting [Araki '75].
- $\zeta_1(\mathbb{1}) = \zeta_2(\mathbb{1}) \Rightarrow S(\zeta_1|\zeta_2) \in [0, +\infty]$, and $S(\zeta_1|\zeta_2) = 0$ iff $\zeta_1 = \zeta_2$.

Setup for Landauer's Principle

The system $\mathcal{S}$

- $\mathcal{O}_{\mathcal{S}} = \mathcal{B}(\mathcal{H}_{\mathcal{S}})$ finite dimensional C^*-algebra.
- Initial state $\rho_{\rm i}(A) = \operatorname{tr}(\rho_i A)$.

The Thermal reservoir $\mathcal{R}$

- C^*-dynamical system $(\mathcal{O}_{\mathcal{R}}, \tau_{\mathcal{R}})$.
- $\tau_{\mathcal{R}}^t = \mathrm{e}^{t\delta_{\mathcal{R}}}$.
- Initial state is a $(\tau_{\mathcal{R}}, \beta)$-KMS state ν_i.
- Liouvillean $L_{\mathcal{R}}$.

Joint system $\mathcal{S} + \mathcal{R}$

- $\mathcal{O} = \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{R}}$.
- $\omega_i = \rho_{\rm i} \otimes \nu_{\rm i}$.
- Inner automorphism $\alpha_U(A) = U^* A U$, for some unitary $U \in \mathcal{O}$.
- State transformation $\omega_{\rm i} \mapsto \omega_f = \omega_{\rm i} \circ \alpha_U$.
- Reference "state" $\eta = \mathbb{1} \otimes \nu_i$.

General form of Landauer's Principle

Set

$$\Delta S = S(\rho_i) - S(\rho_f), \qquad \Delta Q = -\mathrm{i}\omega_i(U^*\delta_{\mathcal{R}}(U))$$

Theorem 1

Assume that $U \in \mathrm{Dom}(\delta_{\mathcal{R}})$.

- $$\beta\Delta Q \geq \Delta S$$
- If the point spectrum of the Liouvillean $L_{\mathcal{R}}$ is finite then equality holds iff $\Delta S = \beta\Delta Q = 0$. In this case $\nu_f = \nu_i$ and ρ_f is unitarily equivalent to ρ_i.

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Remark 2. If $\mathcal{R}$ is confined then $\delta_{\mathcal{R}} = \mathrm{i}[H_{\mathcal{R}}, \cdot\,]$ and hence

$$\Delta Q = \omega_i(\alpha_U(H_{\mathcal{R}}) - H_{\mathcal{R}}) = \omega_f(H_{\mathcal{R}}) - \omega_i(H_{\mathcal{R}})$$

Moreover, the spectrum of $L_{\mathcal{R}}$ is finite $\Rightarrow$ Reeb-Wolf formulation of Landauer's Principle.

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Remark 4. It is an interesting open problem to characterize all reservoirs for which this second part holds.

The entropy balance equation

Theorem (Perturbation of KMS structure) [Araki '73]

Let $(\mathcal{O}, \tau^t = \mathrm{e}^{t\delta})$ be a C^*-dynamical system and ω a (τ, β)-KMS state.

- If $K = K^* \in \mathcal{O}$, then $\delta_K = \delta + \mathrm{i}[K, \cdot\,]$ generates a C^*-dynamical systems $(\mathcal{O}, \tau_K)$.
- There is a continuous map

  $$\mathcal{O} \ni K = K^* \mapsto \omega_K \in \mathcal{N}_\omega$$

  such that ω_K is a (τ_K, β)-KMS state.
- For any positive linear functional ζ

  $$S(\zeta|\omega_K) = S(\zeta|\omega) + \beta\zeta(K) + \log \|\mathrm{e}^{-\beta(L_\omega + \pi_\omega(K))/2}\Omega_\omega\|^2$$

A consequence of which is

**Theorem (Entropy balance)
[Pusz-Woronowicz '78], [Ojima-Hasegawa-Ichiyanagi '88], [Jakšić-P '01]**

If $U \in \mathrm{Dom}(\delta_{\mathcal{R}})$ then

$$S(\omega \circ \alpha_U|\eta) - S(\omega|\eta) = -\mathrm{i}\beta\omega(U^*\delta_{\mathcal{R}}(U))$$

for any state ω on $\mathcal{O}$.

The entropy balance equation

Proof of Theorem 1

Araki's perturbation theorem (with $\tau = \tau_{\mathcal{R}}$, $\omega = \eta$ and $K = -\beta^{-1} \log \rho$) yields

$$S(\zeta|\rho \otimes \nu_i) = S(\zeta|\eta) - \zeta(\log \rho) \tag{2}$$

for any state ρ on $\mathcal{O}_{\mathcal{S}}$ and ζ on $\mathcal{O}$. In particular, with $\rho = \rho_i$ and $\omega = \omega_i$ the LHS of (2) vanishes and

$$S(\omega_i|\eta) = \omega_i(\log \rho_i) = \mathrm{tr}(\rho_i \log \rho_i) = -S(\rho_i)$$

So we can write the entropy balance equation

$$S(\omega_f|\eta) - S(\omega_i|\eta) = -\mathrm{i}\beta\omega_i(U^* \delta_{\mathcal{R}}(U))$$

as

$$S(\rho_i) - S(\rho_f) + S(\omega_f|\eta) + S(\rho_f) = -\mathrm{i}\beta\omega_i(U^* \delta_{\mathcal{R}}(U))$$

which means

$$\Delta S + \sigma = \beta \Delta Q$$

whith the **entropy production** term

$$\sigma = S(\omega_f|\eta) + S(\rho_f)$$

Using again (2) with $\zeta = \omega_f$ and $\rho = \rho_f$ we finally get

$$\sigma = S(\omega_f|\rho_f \otimes \nu_i) \geq 0$$

with equality iff $\omega_f = \rho_f \otimes \nu_i$. The proof of the second part of the theorem relies on the spectral analysis of modular operators ($\Delta_{\omega_f|\omega_i} = \pi_{\omega_i}(U)\Delta_{\omega_i}\pi_{\omega_i}(U)^*$)

Instantaneously switched interactions

We specialize the previous setup to Hamiltonian dynamics.

Let $K = K^* \in \mathrm{Dom}(\delta_{\mathcal{R}})$ and τ_k the dynamics generated by $\delta_K = \delta_{\mathcal{R}} + \mathrm{i}[K, \cdot]$.

Interaction picture

$$\tau_K^t(A) = \tau_{\mathcal{R}}^t(U_k^*(t) A U_K(t))$$

where U_K is a family of unitary elements of $\mathcal{O}$ satisfying

$$\mathrm{i}\partial_t U_K(t) = U_K(t)\tau_{\mathcal{R}}^{-t}(K), \qquad U_K(0) = \mathbb{1}$$

and $U_k(t) \in \mathrm{Dom}(\delta_{\mathcal{R}})$ for all t. Hence, applying our general result to

$$\omega_i \circ \tau_K^t = \omega_i \circ \alpha_{U_K(t)}$$

we get

$$\Delta S(K, t) + \sigma(K, t) = \beta \Delta Q(K, t)$$

with

$$\Delta S(K, t) = S(\rho_i) - S(\rho_K(t)), \qquad \sigma(K, t) = S(\omega_i \circ \tau_K^t | \rho_K(t) \otimes \nu_i), \qquad \rho_K(t) = \omega_i \circ \tau_K^t|_{\mathcal{O}_{\mathcal{S}}}$$

and one easily checks that

$$\Delta Q(K, t) = -\mathrm{i}\omega_i(U_K^*(t)\delta_{\mathcal{R}}(U_K(t))) = \omega_i(K - \tau_K^t(K)) = \int_0^t \omega_i \circ \tau_K^s(\Phi)\, \mathrm{d}s$$

where $\Phi = \delta_K(K) = \delta_{\mathcal{R}}(K)$ is the instantaneous energy flux out of $\mathcal{R}$.

Instantaneously switched interactions – The large time limit

The state transition $\rho_i \to \rho_f = \lim_{t\to\infty} \rho_K(t)$

In the following we fix faithful initial/target state ρ_i/ρ_f, and for simplicity make

Assumption P. ν_i is extremal $(\tau_{\mathcal{R}}, \beta)$-KMS state.

For any $K = K^* \in \mathcal{O}$ there is a unique (τ_K, β)-KMS state $\mu_K \in \mathcal{N}_{\omega_i}$ and $\mathcal{N}_{\mu_K} = \mathcal{N}_{\omega_i}$.
A simple application of the analytic implicit function theorem yields

Proposition 2

Let $V = V^* \in \mathcal{O}$. There exists $\delta > 0$ and a real analytic function

$$]-\delta, \delta[\ni \lambda \to H_\lambda = H_\lambda^* \in \mathcal{O}_{\mathcal{S}}$$

such that $H_0 = -\beta^{-1} \log \rho_f$ and $\mu_{K_\lambda}|_{\mathcal{O}_{\mathcal{S}}} = \rho_f$ for $K_\lambda = H_\lambda + \lambda V$ and any $\lambda \in]-\delta, \delta[$.

Assumption A. There is $\gamma \in]-\delta, \delta[$ such that μ_{K_γ} is mixing.

Instantaneously switched interactions – The large time limit

It follows that

$$\lim_{t\to\infty} \rho_{K_\gamma}(t) = \rho_f$$

$$\Delta S = \lim_{t\to\infty} \Delta S(K_\gamma, t) = S(\rho_i) - S(\rho_f)$$

$$\Delta Q(\gamma) = \lim_{t\to\infty} \Delta Q(K_\gamma, t) = \omega_i(K_\gamma) - \mu_{K_\gamma}(K_\gamma) = -\int_0^\infty \omega_i \circ \tau_{K_\gamma}^s(\Phi)\,\mathrm{d}s$$

and the entropy balance relation yields the existence of

$$\lim_{t\to\infty} \sigma(K, t) = \sigma(\gamma) \geq 0$$

and Landauer's Principle

$$\Delta S + \sigma(\gamma) = \beta \Delta Q(\gamma)$$

for the transition $\rho_i \to \rho_f$.

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The weak-$*$ lower semicontinuity of relative entropy and the KMS structure imply

Proposition 3

$$\sigma(\gamma) \geq S(\mu_{K_\gamma} | \rho_f \otimes \nu_i) > 0$$

The Landauer bound **can not** be saturated by instantaneously switched interactions.

Instantaneously switched interactions – The large time limit

Remark 1. The above analysis extends to W^*-dynamical systems. Unbounded interactions V satisfying the assumptions in [Dereziński-Jakšić-P '03] are also allowed.

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Remark 2. Pure target states $\rho_f = |\psi\rangle\langle\psi|$ are thermodynamically singular and cannot be reached by the action of a reservoir at non-zero temperature. Approximating ρ_f by faithful ρ leads to instability since $\rho \to \rho_f$ implies $S(\rho) \to S(\rho_f) = 0$ and $\sigma(\gamma) \to \infty$ and hence $\Delta Q(\gamma) \to \infty$.

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Remark 3. Assuming that μ_{K_λ} is mixing for all sufficiently small $\lambda \neq 0$ we can let $\lambda \to 0$. Since ΔS is independent of λ and $\lambda \mapsto \Delta Q(\lambda)$ is analytic we deduce that $\lambda \mapsto \sigma(\lambda)$ is also analytic and in the limit $\lambda \to 0$ we get

$$\Delta S + \sigma(0) = \beta \Delta Q(0)$$

where

$$\Delta Q(0) = \omega_f(H_0) - \omega_i(H_0), \qquad \sigma(0) = S(\rho_f|\rho_i)$$

One can check that this limiting form of LP coincide with the LP for the Markovian dynamics emerging from the van Hove weak coupling limit $\lambda \to 0$, $t \sim \lambda^{-2}$ [Lebowitz-Spohn '78].

Adiabaticaly switched interactions

- Let $[0,1] \ni t \mapsto K(t) = K(t)^* \in \mathrm{Dom}(\delta_{\mathcal{R}})$ be a C^2 map on $]0,1[$ with bounded first and second derivatives
- Set $K_T(t) = K(t/T)$

> $\tau^t_{K_T}$ is the non-autonomous dynamics generated by $\delta_{\mathcal{R}} + \mathrm{i}[K_T(t), \cdot\,]$

- Interaction picture

$$\tau_{\mathcal{R}}^{-t} \circ \tau^t_{K_T}(A) = \tau_{\mathcal{R}}^{-t}(\Gamma_{K_T}(t))^* A \tau_{\mathcal{R}}^{-t}(\Gamma_{K_T}(t))$$

$\Gamma_{K_T}(t) \in \mathrm{Dom}(\delta_{\mathcal{R}})$ is the family of unitaries satisfying

$$\mathrm{i}\partial_t \Gamma_{K_T}(t) = \tau^t_{\mathcal{R}}(K_T(t))\Gamma_{K_T}(t), \qquad \Gamma_{K_T}(0) = \mathbb{1}$$

- Since $\omega_i \circ \tau^t_{K_T} = \omega_i \circ \alpha_{\Gamma_{K_T}(t)}$ we can again apply our general result to get

$$\boxed{\Delta S_T + \sigma_T = \beta \Delta Q_T}$$

$$\Delta S_T = S(\rho_i) - S(\rho_T), \qquad \rho_T = \omega_i \circ \tau^T_{K_T}|_{\mathcal{O}_{\mathcal{S}}}$$

$$\Delta Q_T = -\mathrm{i}\omega_i(\Gamma_{K_T}(T)^* \delta_{\mathcal{R}}(\Gamma_{K_T}(T))), \qquad \sigma_T = S(\omega_i \circ \tau^T_{K_T} | \rho_T \otimes \nu_i)$$

- The energy balance is given by

$$\Delta Q_T + \left[\omega_i \circ \tau^T_{K_T}(K_T(T)) - \omega_i(K_T(0))\right] = \int_0^T \omega_i \circ \tau^t_{K_T}(\partial_t K_T(t))\,\mathrm{d}t$$

The adiabatic limit

To deal with the adiabatic limit $T \to \infty$, we replace Assumption A by

Assumption B. For $\gamma \in]0,1[$ the $(\tau_{K(\gamma)}, \beta)$-KMS state $\mu_{K(\gamma)}$ is ergodic for the dynamical system $(\mathcal{O}, \tau_{K(\gamma)})$

Combining the gapless adiabatic theorem of [Avron-Elgart '99], [Teufel '01] and Araki's perturbation theory of KMS states leads to

Theorem 4

Suppose that Assumptions P and B hold. Then one has

$$\lim_{T\to\infty} \|\mu_{K(0)} \circ \tau_{K_T}^{\gamma T} - \mu_{K(\gamma)}\| = 0$$

for all $\gamma \in [0,1]$.

Similar result was obtained and used by [Abou Salem-Fröhlich '05] to analyse quasi-static thermodynamic processes.

The adiabatic limit

According to the above adiabatic theorem, to implement the state transition $\rho_i \to \rho_f$ in the limit $T \to \infty$ it suffices to supplement Assumption B with the boundary conditions

$$K_0 = -\beta^{-1} \log \rho_i, \qquad K_1 = -\beta^{-1} \log \rho_f$$

which imply $\mu_{K_0} = \rho_i \otimes \nu_i$ and $\mu_{K_1} = \rho_f \otimes \nu_i$ so that

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It follows that

$$\Delta S = \lim_{T\to\infty} \Delta S_T = S(\rho_i) - S(\rho_f)$$

The energy balance equation, written as

$$\Delta Q_T = \int_0^1 \omega_i \circ \tau_{K_T}^{\gamma T}(\partial_\gamma K(\gamma))\,\mathrm{d}\gamma - \beta^{-1}\omega_i \circ \tau_{K_T}^{T}(\log \rho_f) + \beta^{-1}\omega_i(\log \rho_i)$$

further gives

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which yields Landauer's Principle

$$\beta\Delta Q = \Delta S + \sigma, \qquad \sigma = \lim_{T\to\infty} \sigma_T = \beta \int_0^1 \mu_{K(\gamma)}(\partial_\gamma K(\gamma))\,\mathrm{d}\gamma \geq 0$$

The adiabatic limit

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Remark 2. Since $\sigma = 0$, the instability observed in the limit $\rho_f \to |\psi\rangle\langle\psi|$ in the case of instantaneous switching does not occur here.

Remark 3. The proof of the above proposition requires modular theory. It is a simple adaptation of the following elementary calculation which holds for finite reservoirs

$$\begin{aligned}\int_0^1 \mu_{K(\gamma)}(\partial_\gamma K(\gamma))\,\mathrm{d}\gamma &= \int_0^1 \frac{\mathrm{tr}\left(\mathrm{e}^{-\beta(H_{\mathcal{R}}+K(\gamma))}\partial_\gamma K(\gamma)\right)}{\mathrm{tr}\left(\mathrm{e}^{-\beta(H_{\mathcal{R}}+K(\gamma))}\right)}\,\mathrm{d}\gamma \\ &= -\frac{1}{\beta}\int_0^1 \partial_\gamma \log\mathrm{tr}\left(\mathrm{e}^{-\beta(H_{\mathcal{R}}+K(\gamma))}\right)\,\mathrm{d}\gamma \\ &= -\frac{1}{\beta}\left(\log\mathrm{tr}(\rho_f\otimes\nu_i) - \log\mathrm{tr}(\rho_i\otimes\nu_i)\right) = 0\end{aligned}$$

Note however that Theorem 4 and existence of $\lim_{T\to\infty}\sigma_T$ can not hold for finite reservoir.

3. Summary

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- The entropy balance relation is a model independent structural identity. It is tautological for confined systems

$$\begin{aligned} S(\omega \circ \alpha_U|\eta) - S(\omega|\eta) &= \mathrm{tr}\left(U\omega U^*(U\log\omega U^* - \log\eta) - \omega(\log\omega - \log\eta)\right) \\ &= \mathrm{tr}\left(\omega(U^*\log\eta U - \log\eta)\right) \\ &= -\beta\mathrm{tr}\left(\omega(U^* H_{\mathcal{R}} U - H_{\mathcal{R}})\right) \end{aligned}$$

  It follows from Araki's perturbation theory of KMS structure for extended systems.
  It plays a central role in the analysis of the second law in open quantum systems.
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- These ergodic properties have been established for various models [Botvich-Malyshev '83, Aizenstadt-Malyshev '87, Jakšić-P '96, Bach-Fröhlich-Sigal '00, Jakšić-P '02, Dereziński-Jakšić '03, Fröhlich-Merkli-Ueltschi '03, Aschbacher-Jakšić-Pautrat-P '06, Jakšić-Ogata-P '06, Merkli-Mück-Sigal '07, de Roeck-Kupianien '11]. Further progress in this direction requires novel ideas and techniques in the study of the Hamiltonian dynamics of extended systems.

Thank you !