Spectral and Scattering Properties of Twisted Waveguides

Spectral Days, CIRM, Luminy

June 10, 2014

Based on the following works:

- Ph. Briet, H. Kovarik, G. Raikov, E. Soccorsi, *Eigenvalue asymptotics in a twisted waveguide*, Commun. P.D.E., **34** (2009), 818–836.

- Ph. Briet, H. Kovarik, G. Raikov, *Scattering in twisted waveguides*, J. Funct. Anal. **266** (2014), 1 – 35.

- G. Raikov, *Spectral asymptotics for waveguides with perturbed periodic twisting*, Preprint, 2014.

1. The Dirichlet Laplacian in a Twisted Waveguide

Let

- $\omega \subset \mathbb{R}^2$ be a bounded domain with C^2-boundary, such that $0 \in \omega$;

- $\Omega := \omega \times \mathbb{R}$;

- $\theta \in C^1(\mathbb{R};\mathbb{R})$, $\theta' \in \mathsf{L}^\infty(\mathbb{R})$.

Introduce the twisted waveguide

$$\Omega_\theta := \{r_\theta(x_3)\mathbf{x},\ \ \mathbf{x} \in \Omega\}$$

where

$$r_\theta(x_3) := \begin{pmatrix} \cos\theta(x_3) & \sin\theta(x_3) & 0 \\ -\sin\theta(x_3) & \cos\theta(x_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define the Dirichlet Laplacian $-\Delta^D$ as the self-adjoint operator generated in $\mathsf{L}^2(\Omega_\theta)$ by the closed quadratic form

$$\tilde{\mathcal{Q}}_\theta[f] := \int_{\Omega_\theta} |\nabla f(\mathbf{x})|^2 \, d\mathbf{x},$$

$$f \in \mathsf{D}(\tilde{\mathcal{Q}}_\theta) := \mathsf{H}_0^1(\Omega_\theta).$$

Let $\mathcal{U} : \mathsf{L}^2(\Omega_\theta) \to \mathsf{L}^2(\Omega)$ be the unitary operator given by

$$(\mathcal{U}f)(\mathbf{x}) = f(r_\theta(x_3)\, \mathbf{x}),\, \mathbf{x} \in \Omega,\, f \in \mathsf{L}^2(\Omega_\theta).$$

Set

$$\nabla_t := (\partial_1, \partial_2), \qquad \Delta_t := \partial_1^2 + \partial_2^2,$$

$$\partial_\varphi := x_1 \partial_2 - x_2 \partial_1.$$

Define the operator $H_{\theta'}$ as the self-adjoint operator generated in $\mathsf{L}^2(\Omega)$ by the closed quadratic form

$$\mathcal{Q}_{\theta'}[f] := \int_{\Omega} (|\nabla_t f|^2 + |\theta'(x_3)\partial_\varphi f + \partial_3 f|^2) d\mathbf{x},$$

$$f \in D(\mathcal{Q}_{\theta'}) := \mathsf{H}_0^1(\Omega).$$

Evidently, $H_{\theta'}$ is strictly positive, and hence invertible, in $\mathsf{L}^2(\Omega)$.

Proposition 1. *Assume that* $\omega \subset \mathbb{R}^2$ *is a bounded domain with boundary* $\partial\omega \in C^2$, *and* $\theta \in C^2(\mathbb{R};\mathbb{R})$ *with* $\theta', \theta'' \in \mathsf{L}^\infty(\mathbb{R})$. *Then the domain of the operator* $H_{\theta'}$ *coincides with* $\mathsf{H}^2(\Omega) \cap \mathsf{H}^1_0(\Omega)$.

The operator H_g acts on its domain as

$$H_{\theta'} = -\Delta_t - (\theta'(x_3)\partial_\varphi + \partial_3)^2.$$

Since

$$\mathcal{Q}_{\theta'}[f] = \tilde{\mathcal{Q}}_\theta[\mathcal{U}^{-1} f], \qquad f \in \mathsf{H}^1_0(\Omega),$$

and $\mathcal{U}$ maps bijectively $\mathsf{H}^1_0(\Omega_\theta)$ onto $\mathsf{H}^1_0(\Omega)$, we have

$$H_{\theta'} = \mathcal{U}(-\Delta^D)\mathcal{U}^{-1},$$

i.e. $-\Delta^D$ is unitarily equivalent to $H_{\theta'}$.

2. Existence and completeness of the wave operators

Theorem 2. *Assume that* $\omega \subset \mathbb{R}^2$ *is a bounded domain with* C^2*-boundary. Let* $\theta_j \in C^2(\mathbb{R};\mathbb{R})$ *with* $\theta'_j, \theta''_j \in \mathsf{L}^\infty(\mathbb{R})$*,* $j = 1, 2$*. Suppose that there exist* $\alpha > 1$ *and* $C \in [0, \infty)$ *such that*

$$|\theta'_1(x) - \theta'_2(x)| \leq C(1 + |x|)^{-\alpha}, \quad x \in \mathbb{R}.$$

Then the operator $H_{\theta'_1}^{-2} - H_{\theta'_2}^{-2}$ *is trace class.*

Corollary 3. *Under the assumptions of Theorem 2 the wave operators*

$$\mathrm{s} - \lim_{t \to \mp\infty} \exp(itH_{\theta'_1}) \exp(-itH_{\theta'_2}) P_{\mathrm{ac}}(H_{\theta'_2})$$

for the operator pair $(H_{\theta'_1}, H_{\theta'_2})$ *exist and are complete. Therefore, the absolutely continuous parts of* $H_{\theta'_1}$ *and* $H_{\theta'_2}$ *are unitarily equivalent, and, in particular,*

$$\sigma_{\mathrm{ac}}(H_{\theta'_1}) = \sigma_{\mathrm{ac}}(H_{\theta'_2}).$$

3. Constant twisting

Assume that there exists a $\beta \in \mathbb{R}$ such that $\theta'(x_3) = \beta$ for every $x_3 \in \mathbb{R}$.

Let $\mathcal{F}$ be the partial Fourier transform with respect to x_3, i.e.

$$(\mathcal{F}u)(x_t, k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx_3} u(x_t, x_3) dx_3$$

for $u \in L^2(\omega; \mathcal{S}(\mathbb{R}))$. We have

$$\mathcal{F} H_\beta \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h_\beta(k) dk$$

where

$$h_\beta(k) := -\Delta_t - (\beta \partial_\varphi + ik)^2, \quad k \in \mathbb{R}.$$

For each $k \in \mathbb{R}$ the spectrum of the operator $h_\beta(k)$ is discrete. Let

$$\left\{E_j(k)\right\}_{j=1}^{\infty} = \left\{E_j(k,\beta)\right\}_{j=1}^{\infty}$$

be the non-decreasing sequence of its eigen-values.

The functions

$$\mathbb{R} \ni k \mapsto E_j(k) \in (0,\infty), \quad j \in \mathbb{N},$$

are continuous piecewise analytic functions.

Moreover,

$$E_j(k) = k^2(1 + o(1)), \quad k \to \pm\infty.$$

Hence,

$$\sigma(H_\beta) = \sigma_{\mathrm{ac}}(H_\beta) = \cup_{j\in\mathbb{N}} E_j(\mathbb{R}) = [\mathcal{E}_0^+, \infty)$$

with

$$\mathcal{E}_0^+ := \min_{k\in\mathbb{R}} E_1(k,\beta).$$

4. Asymptotic completeness of the wave operators for waveguides with perturbed constant twisting

Theorem 4. *Let* $\theta'(x_3) = \beta - \varepsilon(x_3)$*, where* $\beta \in \mathbb{R}$ *and* $\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$*. Assume that there exists* $C \in [0, \infty)$ *such that*

$$|\varepsilon(x)| + |\varepsilon'(x)| \leq C(1 + x^2)^{-1}, \quad x \in \mathbb{R}.$$

Then there exists a locally finite (hence, discrete) set $\mathcal{T} \subset \mathbb{R}$ *such that:*
(i) *Any compact subinterval of* $\mathbb{R} \setminus \mathcal{T}$ *contains at most finitely many eigenvalues of* $H_{\theta'}$*, each having a finite multiplicity;*
(ii) *The singular continuous spectrum of* $H_{\theta'}$ *is empty.*

Corollary 5. *Under the assumptions of Theorem 4 the wave operators for the operator pair* $(H_{\theta'}, H_\beta)$ *exist and are asymptotically complete.*

Main ingredient of the proof of Theorem 4:
Mourre estimates

More precisely, for any $E \in \mathbb{R} \setminus \mathcal{T}$ there exists $\delta > 0$ such that the strict Mourre estimates

$$\mathbf{1}_{(E-\delta,E+\delta)}(H_\beta)[H_\beta, iA]\mathbf{1}_{(E-\delta,E+\delta)}(H_\beta) \geq C\mathbf{1}_{(E-\delta,E+\delta)}(H_\beta) \tag{1}$$

hold true with an appropriate conjugate operator A and a constant $C > 0$. If, moreover, ε satisfies the assumptions of Theorem 4, then the non strict Mourre estimates

$$\mathbf{1}_{(E-\delta,E+\delta)}(H_{\theta'})[H_{\theta'}, iA]\mathbf{1}_{(E-\delta,E+\delta)}(H_{\theta'}) \geq C'\mathbf{1}_{(E-\delta,E+\delta)}(H_{\theta'}) + K \tag{2}$$

hold true with same A as in (1), a constant $C' > 0$, and a compact operator K.

Idea of the construction of the conjugate operator A:

We have

$$A = \mathcal{F}^* \left(\mathbf{1}_\omega \otimes a \right) \mathcal{F}$$

where

$$a := \frac{i}{2} \left(\gamma \frac{d}{dk} + \frac{d}{dk} \gamma \right),$$

and $\gamma \in C_0^\infty(\mathbb{R};\mathbb{R})$ is an appropriate function.

4. Eigenvalue asymptotics for waveguides with perturbed periodic twisting

Assume that $\beta = \overline{\beta} \in C(\mathbb{T})$ where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Set $\mathbb{T}^* := \mathbb{R}/\mathbb{Z}$. Introduce the unitary Floquet-Bloch operator $\Phi : L^2(\Omega) \to L^2(\omega \times \mathbb{T} \times \mathbb{T}^*)$,

$$(\Phi u)(x_t, x_3, k) :=$$

$$\sum_{\ell \in \mathbb{Z}} e^{-ik(x_3+2\pi\ell)} u(x_t, x_3 + 2\pi\ell),$$

$x_t \in \omega$, $x_3 \in \mathbb{T}$, $k \in \mathbb{T}^*$. Then

$$\Phi H_\beta \Phi^* = \int_{\mathbb{T}^*}^{\oplus} h_\beta(k) dk$$

where

$$h_\beta(k) := -\Delta_t - (\beta \partial_\varphi + \partial_3 + ik)^2, \quad k \in \mathbb{T}^*,$$

is self-adjoint in $\mathrm{L}^2(\omega \times \mathbb{T})$. Let $\{E_\ell(k)\}_{\ell \in \mathbb{N}}$ be the non-decreasing sequence of the eigenvalues of $h_\beta(k)$, $k \in \mathbb{T}^*$. We have

$$\sigma(H_\beta) = \bigcup_{\ell \in \mathbb{N}} E_\ell(\mathbb{T}^*).$$

Set

$$\mathcal{E}_0^+ := \inf \sigma(H_\beta) = \min_{k \in \mathbb{T}^*} E_1(k).$$

In $\sigma(H_\beta)$ there always exists the semi-bounded gap $(-\infty, \mathcal{E}_0^+)$. It is possible also to have bounded open gaps $(\mathcal{E}_j^-, \mathcal{E}_j^+)$, $j = 1, \ldots, J$. One way to see this, is to consider thin twisted waveguides. Namely, for $\ell > 0$ set

$$\omega_\ell := \{x_t \in \mathbb{R}^2 \,|\, \ell^{-1} x_t \in \omega\}, \quad \Omega_\ell := \omega_\ell \times \mathbb{R}.$$

Let $H_{\beta,\ell}$ be the Dirichlet Hamiltonian in a waveguide with $\theta' = \beta$, self-adjoint in $\mathrm{L}^2(\Omega_\ell)$. Then, under appropriate conditions, the operator $H_{\beta,\ell} - \ell^{-2}\lambda_1$ converges in a suitable sense as $\ell \downarrow 0$, to the operator

$$-\frac{d^2}{dx^2} + \|\partial_\varphi \Psi_1\|^2_{\mathrm{L}^2(\omega)} \beta(x)^2, \quad x \in \mathbb{R},$$

self-adjoint in $\mathrm{L}^2(\mathbb{R})$. Here, λ_1 is the first eigenvalue of the Dirichlet Lalpacian $-\Delta_t$, self-adjoint in $\mathrm{L}^2(\omega)$, and Ψ_1 is the corresponding eigenfunction, normalized in $\mathrm{L}^2(\omega)$.

Then we have

$$\mathbb{R} \setminus \sigma(H_\beta) = \bigcup_{j=0}^{J} \left(\mathcal{E}_j^-, \mathcal{E}_j^+\right)$$

with $\mathcal{E}_0^- := -\infty$. The value $\mathcal{E}_j^-$, $j \geq 1$, (resp, $\mathcal{E}_j^+$, $j \geq 0$) coincides with the maximal (resp., minimal) value of some band function E_ℓ.

Definition: The edge point $\mathcal{E}_j^\pm$ is regular if:
(i) There exists a unique band function $E_{\ell(j)}^\pm$ which takes the value $\mathcal{E}_j^\pm$.
(ii) The function $E_{\ell(j)}^\pm$ takes the value $\mathcal{E}_j^\pm$ at finitely many points $k_{j,m}^\pm$, $m = 1, \ldots, M_j^\pm$.
(iii) We have

$$\mu_{j,m}^\pm := \pm \frac{1}{2} \frac{d^2 E_{\ell(j)}^\pm}{dk^2}(k_{j,m}^\pm) > 0, \quad m = 1, \ldots, M_j^\pm.$$

If conditions (i) and (ii) hold true, then $E^{\pm}_{\ell(j)}$ is analytic in a vicinity of each point $k^{\pm}_{j,m}$, i.e. there exists a $\delta > 0$ such that the intervals

$$\mathcal{I}^{\pm}_{j,m} = \left[k^{\pm}_{j,m} - \delta, k^{\pm}_{j,m} + \delta\right], \quad m = 1, \ldots, M^{\pm}_{j},$$

are disjoint, and $E^{\pm}_{\ell(j)}$ is real-analytic on them.

Proposition 6. *If $\beta \in C^2(\mathbb{T})$, the edge $\mathcal{E}^{+}_{0}$ is regular, $M^{+}_{0} = 1$, and $k^{+}_{0,1} = 0$.*

Introduce the eigenfunctions

$$\psi^{\pm}_{j}(\mathbf{x}; k), \mathbf{x} \in \omega \times \mathbb{T},\ k \in \mathcal{I}^{\pm}_{j} := \bigcup_{m=1}^{M^{\pm}_{j}} \mathcal{I}^{\pm}_{j,m},$$

such that

$$h_{\beta}(k)\psi^{\pm}_{j}(\cdot; k) = E^{\pm}_{\ell(j)}\psi^{\pm}_{j}(\cdot; k),$$

$$\int_{\omega}\int_{\mathbb{T}}\left|\psi^{\pm}_{j}(x_t, x_3; k)\right|^2 dx_3 dx_t = 1, \quad k \in \mathcal{I}^{\pm}_{j},$$

and the mapping

$$\mathcal{I}^{\pm}_{j} \ni k \mapsto \psi^{\pm}_{j}(\cdot; k) \in D(H_{\beta})$$

is analytic.

Assume

$$\beta \in C(\mathbb{T}), \quad \varepsilon \in C(\mathbb{R}), \quad \lim_{|x|\to\infty} \varepsilon(x) = 0.$$

Then the operator $H_\beta^{-1} - H_{\beta-\varepsilon}^{-1}$ is compact, and $\sigma_{\text{ess}}(H_\beta) = \sigma_{\text{ess}}(H_{\beta-\varepsilon})$. Therefore,

$$\mathbb{R} \setminus \sigma_{\text{ess}}(H_{\beta-\varepsilon}) = \bigcup_{j=0}^{J} \left(\mathcal{E}_j^-, \mathcal{E}_j^+\right).$$

Let $T = T^*$,

$$N_{\mathcal{I}}(T) := \text{rank}\, \mathbb{1}_{\mathcal{I}}(T)$$

where $\mathbb{1}_{\mathcal{I}}(T)$ is the spectral projection of T corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Put

$$\mathcal{N}_0^+(\lambda) = N_{(-\infty, \mathcal{E}_0^+ - \lambda)}(H_{\beta-\varepsilon}), \quad \lambda > 0.$$

Fix $\mathcal{E} \in \left(\mathcal{E}_j^-, \mathcal{E}_j^+\right)$, $j \geq 1$, and set

$$\mathcal{N}_j^-(\lambda) = N_{(\mathcal{E}_j^- + \lambda, \mathcal{E})}(H_{\beta-\varepsilon}), \quad \lambda \in (0, \mathcal{E} - \mathcal{E}_j^-),$$

$$\mathcal{N}_j^+(\lambda) = N_{(\mathcal{E}, \mathcal{E}_j^+ - \lambda)}(H_{\beta-\varepsilon}), \quad \lambda \in (0, \mathcal{E}_j^+ - \mathcal{E}).$$

Assume that the edge point $\mathcal{E}_j^{\pm}$ is regular. For $x_3 \in \mathbb{T}$ and $m = 1, \ldots, M_j^{\pm}$, introduce the functions

$$\eta_{j,m}^{\pm}(x_3) :=$$

$$4\pi \mathrm{Re} \int_{\omega} \overline{\partial_{\varphi}\psi_j^{\pm}(x_t, x_3; k_{j,m}^{\pm})}$$

$$\left(\beta(x_3)\partial_{\varphi} + \partial_3 + ik_{j,m}^{\pm}\right) \psi_j^{\pm}(x_t, x_3; k_{j,m}^{\pm})\, dx_t,$$

and their mean values

$$\langle \eta_{j,m}^{\pm} \rangle := \frac{1}{2\pi} \int_{\mathbb{T}} \eta_{j,m}^{\pm}(x) dx.$$

For $n \in \mathbb{N}$ and $\alpha > 0$ set

$$\mathcal{S}_{n,\alpha}(\mathbb{R}) := \{u = \overline{u} \in C^n(\mathbb{R}) \mid$$

$$|u^{(\ell)}(x)| \leq c_{\ell}(1 + |x|)^{-\alpha-\ell},\ x \in \mathbb{R},\ \ell = 0, \ldots, n\Big\}.$$

$$\mathcal{S}_{n,\alpha}^{+}(\mathbb{R}) := \{u \in \mathcal{S}_{n,\alpha}(\mathbb{R}) \mid$$

$$\exists C > 0, R > 0 \,\text{such that}\, u(x) \geq C|x|^{-\alpha},\ |x| \geq R\}.$$

Informally, our two next theorems will say that

$$\mathcal{N}_j^{\pm}(\lambda) \sim$$

$$\sum_{m=1}^{M_j^{\pm}} N_{(-\infty,-\lambda)}\left(-\mu_{j,m}^{\pm}\frac{d^2}{dx^2} \mp \langle\eta_{j,m}^{\pm}\rangle\varepsilon(x)\right),$$

as $\lambda \downarrow 0$. Hence, at that moment, the operator

$$\bigoplus_{m=1}^{M_j^{\pm}}\left(-\mu_{j,m}^{\pm}\frac{d^2}{dx^2} \mp \langle\eta_{j,m}^{\pm}\rangle\varepsilon(x)\right) \qquad (3)$$

self-adjoint in $\mathrm{L}^2(\mathbb{R})$, can be considered as the *effective Hamiltonian* which models the behaviour of the discrete spectrum of $H_{\beta-\varepsilon}$ near the regular edge point $\mathcal{E}_j^{\pm}$ of $\sigma(H_\beta)$.

Theorem 7. *Let $\beta \in C^4(\mathbb{T})$, and $\left(\mathcal{E}_j^-, \mathcal{E}_j^+\right)$, $j \geq 0$, be an open gap in $\sigma(H_\beta)$. Assume that the edge point $\mathcal{E}_j^\pm$ is regular.*

(i) *Let $\alpha \in (0,2)$, $\varepsilon \in \mathcal{S}_{4,\alpha}^+(\mathbb{R})$. If there exists at least one m such that $\pm\langle \eta_{j,m}^\pm \rangle > 0$, we have*

$$\mathcal{N}_j^\pm(\lambda) =$$

$$\sum_{m=1}^{M_j^\pm} \frac{1}{\pi\sqrt{\mu_{j,m}^\pm}} \int_{\mathbb{R}} \left(\pm\langle \eta_{j,m}^\pm \rangle \varepsilon(x) - \lambda\right)_+^{1/2} dx \, (1+o(1))$$

$$\asymp \lambda^{\frac{1}{2}-\frac{1}{\alpha}},$$

as $\lambda \downarrow 0$. If, on the contrary, $\pm\langle \eta_{j,m}^\pm \rangle < 0$ for all m, then

$$\mathcal{N}_j^\pm(\lambda) = O(1), \quad \lambda \downarrow 0. \qquad (4)$$

(ii) *Let $\alpha \in (0,2)$, $\pm\langle \eta_{j,m}^\pm \rangle \leq 0$ for all m, and $\langle \eta_{j,m}^\pm \rangle = 0$ for some m. Suppose that $\varepsilon \in \mathcal{S}_{4,\alpha}(\mathbb{R})$. Then for each $\kappa > 0$ we have*

$$\mathcal{N}_j(\lambda) = O(\lambda^{\frac{1}{2}(1-\frac{1}{\alpha})-\kappa}), \quad \lambda \downarrow 0,$$

if $\alpha \in (0,1]$, while (4) holds true if $\alpha \in (1,2)$.

(iii) *Let* $\alpha = 2$. *Suppose that and there exists* $L \in \mathbb{R}$ *such that*

$$\lim_{|x| \to \infty} x^2 \varepsilon(x) = L.$$

Then

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1} \mathcal{N}_j^{\pm}(\lambda) = \frac{1}{\pi} \sum_{m=1}^{M_j^{\pm}} \left(\frac{\pm \langle \eta_{j,m}^{\pm} \rangle L}{\mu_{j,m}^{\pm}} - \frac{1}{4} \right)_+^{1/2}.$$

If, moreover, $\pm 4 \langle \eta_{j,m}^{\pm} \rangle L < \mu_{j,m}^{\pm}$ for all m, *then* (4) *holds true.*

(iv) Let $\alpha > 2$. Then (4) holds true again.

Idea of the proof of Theorem 7

An effective Hamiltonian, more informative than the one defined in (3), is the operator

$$\bigoplus_{m=1}^{M_j^{\pm}} \left(-\mu_{j,m}^{\pm} \frac{d^2}{dx^2} \mp \eta_{j,m}^{\pm}(x)\varepsilon(x) \right),$$

which involves the periodic functions $\eta_{j,m}^{\pm}$ instead of their mean values $\langle \eta_{j,m}^{\pm} \rangle$.

Basically, we have to deal with the asymptotic behaviour of the discrete spectrum of the operator

$$\mathcal{H}_{\text{eff}} := -\frac{d^2}{dx^2} - \eta(x)\varepsilon(x), \quad x \in \mathbb{R},$$

self-adjoint in $\mathsf{L}^2(\mathbb{R})$. Here $\varepsilon \geq 0$ decays regularly at infinity, but η is a 2π-periodic function which generically is not constant, so that the decay of the product $\eta\varepsilon$ is not regular, In the following proposition we summarize the eigenvalue asymptotics for $\mathcal{H}_{\text{eff}}$.

Proposition 8. *Let* $\eta \in C^4(\mathbb{T})$, $\varepsilon \in \mathcal{S}_{4,\alpha}(\mathbb{R})$, $\alpha > 0$.

(i) *Let* $\alpha \in (0,2)$, $\varepsilon \in \mathcal{S}^+_{4,\alpha}(\mathbb{R})$. *If* $\langle \eta \rangle > 0$, *then*

$$N_{(-\infty,-\lambda)}(\mathcal{H}_{\text{eff}}) =$$

$$\frac{1}{\pi}\int_{\mathbb{R}} (\langle \eta \rangle \varepsilon(x) - \lambda)_+^{1/2}\, dx\, (1 + o(1)) \asymp \lambda^{\frac{1}{2}-\frac{1}{\alpha}}$$

as $\lambda \downarrow 0$. *If, on the contrary,* $\langle \eta \rangle < 0$, *then*

$$N_{(-\infty,-\lambda)}(\mathcal{H}_{\text{eff}}) = O(1), \quad \lambda \downarrow 0, \qquad (5)$$

(ii) *Let* $\alpha \in (0,2)$, $\langle \eta \rangle = 0$. *Then,*

$$N_{(-\infty,\lambda)}(\mathcal{H}_{\text{eff}}) = O(\lambda^{\frac{1}{2}(1-\frac{1}{\alpha})-\kappa}),\ \lambda \downarrow 0,\ \kappa > 0,$$

if $\alpha \in (0,1]$, *while* (5) *holds true if* $\alpha \in (1,2)$.

(iii) *Let* $\alpha = 2$. *Assume that*

$$\lim_{|x|\to\infty} x^2 \varepsilon(x) = L \in \mathbb{R}.$$

Then,

$$\lim_{\lambda \downarrow 0} |\ln \lambda|^{-1} N_{(-\infty,-\lambda)}(\mathcal{H}_{\text{eff}}) = \frac{1}{\pi}\left(\langle \eta \rangle L - \frac{1}{4}\right)_+^{1/2}.$$

If, moreover, $4\langle \eta \rangle L < 1$, *then* (5) *holds true.*

(iv) *Let* $\alpha > 2$. *Then* (4) *holds true again.*