Magnetic WKB Constructions

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Journées Spectrales

10 juin 2014

IRMAR

This talk is based on the eponymous joint work with

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Once upon a time...

This talk is devoted to self-adjoint realizations of the magnetic Laplacian $(-i\hbar\nabla + \mathbf{A})^2$ and their spectra in the semiclassical limit $\hbar \to 0$.

Some motivations

1. The lowest eigenvalue $\lambda_1(\hbar)$ of the magnetic Laplacian is involved in the theories of superconductivity and liquid crystals.
2. The study of $\lambda_n(\hbar)$ has an interest in itself. Indeed we would like to understand the analogy between the pure electric Laplacian $-\hbar^2\Delta + V$ and the pure magnetic Laplacian. For instance, it is well-known, that if the minimum of (smooth) V is unique and non-degenerate, the n-th eigenvalue $\lambda_n(\hbar)$ admits the following expansion:
$$\lambda_n(\hbar) = V(x_{\min}) + \mu_n\hbar + o(\hbar),$$
where μ_n is the n-th eigenvalue of $D_x^2 + \text{Hess}_{x_{\min}} V(x)$. Is there an analog in the pure magnetic case?

Some reviews

To get an overview of this problem, one may read

1. the book by Fournais and Helffer,
   Spectral methods in Surface Superconductivity [FH10],
2. the survey by Helffer and Kordyukov,
   Semiclassical spectral asymptotics for a magnetic Schrödinger operator with non-vanishing magnetic field [HK14],
3. or the book by Raymond,
   Little Magnetic Book [Ray14].

Some references

Many results about the asymptotic expansions of $\lambda_1(\hbar)$ have been obtained in the last fifteen years.

1. When the magnetic field is constant:
   1. [Erd96, BH97, BPT98, BS98, dPFS00] (2D, disk),
   2. [HM01] (2D, smooth Ω),
   3. [HM04] (3D, smooth Ω),
   4. [FP11] (3D, ball),
   5. [Bon05] (2D, corners).
2. When the magnetic field is variable:
   1. [LP99, Ray09] (2D, smooth Ω, non vanishing field),
   2. [Mon95, HM96, PK02, HK09, HK12] (2D, smooth Ω, vanishing field),
   3. [LP00, Ray10, HK13a] (3D, smooth Ω, non vanishing field),
   4. [Bon05, BND06, BNF07] (2D, non vanishing field, corners),
   5. [BNDP13] (3D, non vanishing field, corners).

What about $\lambda_2(\hbar)$?

Is $\lambda_1(\hbar)$ simple?

Spectral splitting

There are only few cases when the simplicity of the $\lambda_1(\hbar)$ is established. This problem is related to the approximation of the eigenfunctions in the semiclassical limit: it is also rare to have an asymptotic expansion of the first eigenfunctions.
A reason for this is that, in the above mentioned papers, the spirit of the analysis is essentially variational. It is based on:

1. a construction of appropriate test functions for the first Rayleigh quotient,
2. a reduction, through a space partition of unity and estimates of Agmon type, to local models whose spectrum is known.

Spectral splitting

Let us mention some cases when the simplicity of the lowest eigenvalues is established:

1. the case with corners (at least when a corner is small enough) in dimension two [BND06] and in dimension three [BNR13, BNR14] (case of the cone),
2. a generic variable magnetic field in dimension two (without boundary) [HK11] or three (with smooth boundary) [Ray12], by using polynomial estimates in the phase space,
3. a constant magnetic field in a smooth bounded domain carrying a Neumann condition [FH06], by analyzing the microlocalization of the eigenfunctions and a Grushin reduction.

Hidden "Adiabatic" Limits

In the last two years, with Nicolas Dombrowski, Nicolas Popoff and San Vũ Ngọc, we tried to understand the "multiple scales" induced by the magnetic Laplacian under various (and generic) assumptions:

1. when the field is variable and with a Neumann boundary in dimension two [Ray13b],
2. when the field vanishes along a curve in dimension two [DR13],
3. when the field is constant and when the boundary contains an edge in dimension three [PR13],
4. when the field is variable in dimension two [RVuN13].

We have proved that, in all these situations, the magnetic Laplacian is microlocally and unitarily equivalent to an pure electric Laplacian in an "adiabatic form".

Despite of the apparent variety of the geometric situations, we were always able to exhibit the same structure:

1. the eigenvalues are simple,
2. $\lambda_n(\hbar)$ admit an asymptotic expansion at any order modulo $\mathcal{O}(\hbar^\infty)$,
3. so does the corresponding eigenfunction.

Note that, in [RVuN13], up to paying the price of the symplectic geometry and of the Egorov theorem, we are even able to produce an effective Hamiltonian (Birkhoff normal form) which provides a description of all the eigenvalues below some energy level $C_0\hbar$ (and not only the expansion of an eigenvalue of given order). See also a close result of Helffer and Kordyukov [HK13b] where they construct a Grushin problem.

The moral of [Ray13b, DR13, PR13, RVuN13]

Theorem

In generic situations, the spectral analysis of the magnetic Laplacian can be reduced, in the semiclassical limit, to a (microlocal) adiabatic problem.

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Theorem

In generic situations, the spectral analysis of the magnetic Laplacian can be reduced, in the semiclassical limit, to a (microlocal) adiabatic problem.

We will strongly exploit this idea to answer the main question in this talk:

"Do the magnetic eigenfunctions admit WKB expansions?"

Wentzel, Kramers, Brillouin, $\sim$ 1926

In dieser Note soll eine Methode entwickelt werden, die Eigenwertprobleme der Schrödingerschen "Wellenmechanik" durch sukzessive Approximation von dem Grenzfall der klassischen Mechanik (bzw. der früheren Quantentheorie) aus zu lösen.

G. Wentzel

Die Arbeit behandelt eine Methode zur annähernden Lösung des Schrödingerschen Eigenwert- und Eigenfunktionproblems für ein willkürliches System von einem Freiheitsgrad.

H. Kramers

Je veux montrer que l'équation de Schrödinger peut être résolue par approximations successives, la première approximation redonnant l'ancienne mécanique quantique.

L. Brillouin

A magnetic dream

A magnetic dream

1. As far as we know, it was never proved that the first eigenfunctions of a pure magnetic Laplacian were in a WKB form. When there is an additional electric potential, the WKB expansions are possible as we can see in [HS87] and [MS99].

A magnetic dream

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2. Let me hide for the moment the true motivation of this adventure.

And when you have reached the mountain top, then you shall begin to climb.

The Prophet (1923), Khalil Gibran

A partially semiclassical magnetic Laplacian

We would like to analyze the self-adjoint operators on $\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$ of the following type

$$\mathfrak{L}_h = (hD_s + A_1(s,t))^2 + (D_t + A_2(s,t))^2,$$

where A_1 and A_2 are polynomial, $D = -i\nabla$.

Electric Born-Oppenheimer approximation

The problem of considering partial semiclassical problems appears for instance in the context of [CDS81, Mar89, KMSW92] (see also [Mar07]) where the main issue is to approximate the eigenpairs of operators with electrical potentials in the form:

$$-h^2\Delta_s - \Delta_t + V(s,t).$$

The main idea, due to Born and Oppenheimer in [BO27], is to replace, for fixed s, the operator $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$. Then we are led to consider for instance the reduced operator (called Born-Oppenheimer approximation)

$$-h^2\Delta_s + \mu_1(s),$$

and to apply the semiclassical techniques *à la* Helffer-Sjöstrand [HS84, HS85].

Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the electro-magnetic Laplacian acting on $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\mathcal{M}_{x,\xi} = (D_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2.$$

Denoting by $\mu(x, \xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator:

$$\mu(s, hD_s).$$

Assumption 1

Assumption

- *The family* $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ *is analytic of type (B) in the sense of Kato [Kat66, Chapter VII].*
- *For all* $(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m$*, the bottom of the spectrum of* $\mathcal{M}_{x,\xi}$ *is a simple eigenvalue denoted by* $\mu(x,\xi)$ *(in particular it is an analytic function) and associated with a* L^2*-normalized eigenfunction* $u_{x,\xi}\in\mathcal{S}(\mathbb{R}^n)$ *which also analytically depends on* (x,ξ)*.*
- *The function* μ *admits a unique and non degenerate minimum* μ_0 *at point denoted by* (x_0,ξ_0) *and we have* $\liminf_{|x|+|\xi|\to+\infty}\mu(x,\xi)>\mu_0$*.*
- *The family* $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ *can be analytically extended in a complex neighborhood of* (x_0,ξ_0)*.*

Assumption 2

Assumption

Under Assumption 1, let us denote by Hess $\mu(x_0, \xi_0)$ *the Hessian matrix of* μ *at* (x_0, ξ_0). *We assume that the spectrum of the operator* Hess $\mu(x_0, \xi_0)(\sigma, D_\sigma)$ *is simple.*

Assumption 3

Assumption

For $R \geq 0$, we let $\Omega_R = \mathbb{R}^{m+n} \setminus \overline{\mathcal{B}(0,R)}$. We denote by $\mathfrak{L}_h^{\mathrm{Dir},\Omega_R}$ the Dirichlet realization on Ω_R of $(D_t + A_2(s,t))^2 + (hD_s + A_1(s,t))^2$. We assume that there exist $R_0 \geq 0$, $h_0 > 0$ and $\mu_0^* > \mu_0$ such that for all $h \in (0, h_0)$, the first eigenvalue of $\mathfrak{L}_h^{\mathrm{Dir},\Omega_{R_0}}$ satisfies:

$$\lambda_1^{\mathrm{Dir},\Omega_{R_0}}(h) \geq \mu_0^*.$$

By using the Persson's theorem (see [Per60]), we have the following proposition.

Proposition

Under Assumption 3, there exists $h_0 > 0$ *such that for all* $h \in (0, h_0)$*:*

$$\inf \mathrm{sp}_{\mathrm{ess}}(\mathfrak{L}_h) \geq \mu_0^*.$$

Asymptotic expansions of $\lambda_n(h)$

Theorem

Let us assume Assumptions 1, 2 and 3. For all $n \geq 1$, *there exists* $h_0 > 0$ *such that for all* $h \in (0, h_0)$ *the n-th eigenvalue of* $\mathfrak{L}_h$ *exists and satisfies*

$$\lambda_n(h) = \lambda_{n,0} + \lambda_{n,1} h + \mathcal{O}(h^{3/2}),$$

where $\lambda_{n,0} = \mu_0$ *and* $\lambda_{n,1}$ *is the n-th eigenvalue of* $\frac{1}{2}\text{Hess}_{x_0,\xi_0}\,\mu(\sigma, D_\sigma)$.

Scheme of the proof

The proof of the last theorem is divided into two main steps.

1. The first step is to construct quasimodes as formal series expansions and to apply the spectral theorem. In order to succeed we establish Feynman-Hellmann formulas with multiple parameters which are consequences of the perturbation theory of Kato.

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   2. We establish microlocal polynomial estimates (in the spirit of [Ray12] and also [HK11]) for the eigenfunctions.
   3. We use a Feshbach-Grushin type reduction.

Flavor of the proof

Let us discuss the coherent states part.

Let us recall the formalism of coherent states. We define

$$g_0(\sigma) = \pi^{-m/4} \mathrm{e}^{-|\sigma|^2/2},$$

and the usual creation and annihilation operators

$$\mathfrak{a}_j = \frac{1}{\sqrt{2}}(\sigma_j + \partial_{\sigma_j}), \qquad \mathfrak{a}_j^* = \frac{1}{\sqrt{2}}(\sigma_j - \partial_{\sigma_j}),$$

which satisfy the commutator relations

$$[\mathfrak{a}_j, \mathfrak{a}_j^*] = 1, \qquad [\mathfrak{a}_j, \mathfrak{a}_k^*] = 0 \quad \text{if } k \neq j.$$

We notice that

$$\sigma_j = \frac{1}{\sqrt{2}}(\mathfrak{a}_j + \mathfrak{a}_j^*), \qquad \partial_{\sigma_j} = \frac{1}{\sqrt{2}}(\mathfrak{a}_j - \mathfrak{a}_j^*), \qquad \mathfrak{a}_j\mathfrak{a}_j^* = \frac{1}{2}(D_{\sigma_j}^2 + \sigma_j^2 + 1).$$

For $(u, p) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the coherent state

$$f_{u,p}(\sigma) = \mathrm{e}^{ip\cdot\sigma} g_0(\sigma - u),$$

and the associated projection, defined for $\psi \in \mathrm{L}^2(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$\Pi_{u,p}\psi = \langle \psi, f_{u,p} \rangle_{\mathrm{L}^2(\mathbb{R}^m, \mathrm{d}\sigma)} f_{u,p} = \psi_{u,p} f_{u,p},$$

which satisfies

$$\psi = \int_{\mathbb{R}^{2m}} \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{2m}} |\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p\,\mathrm{d}\tau.$$

We recall that

$$(\mathfrak{a}_j)^\ell(\mathfrak{a}_k^*)^q\psi = \int_{\mathbb{R}^{2m}} \left(\frac{u_j + ip_j}{\sqrt{2}}\right)^\ell \left(\frac{u_k - ip_k}{\sqrt{2}}\right)^q \Pi_{u,p}\psi \, \mathrm{d}u \, \mathrm{d}p.$$

The rescaled operator ($s = x_0 + h^{1/2}\sigma,\ t = \tau$) is

$$\mathcal{L}_h = \left(D_\tau + A_2(x_0 + h^{1/2}\sigma, \tau)\right)^2 + \left(\xi_0 + h^{1/2} D_\sigma + A_1(x_0 + h^{1/2}\sigma, \tau)\right)^2$$

and

$$\mathcal{L}_h = \mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2 + \ldots + h^{M/2}\mathcal{L}_M.$$

If we write the (anti-)Wick ordered operator, we get

$$\mathcal{L}_h = \underbrace{\mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2^{\mathsf{W}} + \ldots + (h^{1/2})^M \mathcal{L}_M^{\mathsf{W}}}_{\mathcal{L}_h^{\mathsf{W}}} + \underbrace{hR_2 + \ldots + (h^{1/2})^M R_M}_{\mathcal{R}_h},$$

where the R_d are the remainders in the Wick ordering and satisfy, for $d \geq 2$,

$$h^{d/2}R_d = h^{d/2}\mathcal{O}_{d-2}(\sigma, D_\sigma),$$

where the notation $\mathcal{O}_d(\sigma, D_\sigma)$ stands for a polynomial operator with total degree in (σ, D_σ) less than d. We recall that

$$\mathcal{L}_h^{\mathsf{W}} = \int_{\mathbb{R}^{2m}} \mathcal{M}_{x_0+h^{1/2}u, \xi_0+h^{1/2}p} \, \mathrm{d}u \, \mathrm{d}p.$$

The most important part of the analysis is then to prove that the remainders are actually small when acting on the eigenfunctions. For that purpose, one needs to use rough pseudo-differential estimates related to some semiclassical Agmon-Persson estimates as well as a precise control of the eigenfunctions in suitable norms (by estimating commutators with the creation and annihilation operators) which are consequences of the following lemma (generalized "IMS" formula, see [Ray13a]).

Lemma ("Localization" of P^2 with respect to $\mathfrak{A}$)

Let H be a Hilbert space and P and $\mathfrak{A}$ be two unbounded operators defined on a domain $\mathsf{D} \subset \mathsf{H}$. We assume that P is symmetric and that $P(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}^*(\mathsf{D}) \subset \mathsf{D}$. Then, for $\psi \in \mathsf{D}$, we have

$$\Re\langle P^2\psi, \mathfrak{A}\mathfrak{A}^*\psi\rangle = \|P(\mathfrak{A}^*\psi)\|^2 - \|[\mathfrak{A}^*, P]\psi\|^2 + \Re\langle P\psi, [[P,\mathfrak{A}],\mathfrak{A}^*]\psi\rangle + \Re\left(\langle P\psi, \mathfrak{A}^*[P,\mathfrak{A}]\psi\rangle - \overline{\langle P\psi, \mathfrak{A}[P,\mathfrak{A}^*]\psi\rangle}\right).$$

The main message of this talk

Theorem

In generic situations, the magnetic eigenfunctions are in a WKB form.

We reduce here our study to the case when $A_2 = 0$. We therefore focus now on operators of the form

$$\mathfrak{L}_h = D_t^2 + (hD_s + A_1(s,t))^2.$$

Theorem

Under Assumptions 1, 2 and 3, there exist a function $\Phi = \Phi(s)$ *defined in a neighborhood* $\mathcal{V}$ *of* x_0 *with* $\Re \mathrm{Hess}\, \Phi(x_0) > 0$ *and, for any* $n \geq 1$*, a sequence of real numbers* $(\lambda_{n,j})_{j\geq 0}$ *such that*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^j,$$

with $\lambda_{n,0} = \mu_0$*. Besides there exists a formal series of smooth functions on* $\mathcal{V} \times \mathbb{R}^n_t$*,*

$$a_n(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^j, \quad \text{with } a_{n,0} \neq 0 \text{ such that}$$

$$(\mathfrak{L}_h - \lambda_n(h)) \left(a_n(.;h) \mathrm{e}^{-\Phi/h} \right) = \mathcal{O}\left(h^\infty\right) \mathrm{e}^{-\Phi/h}.$$

In addition, there exists $c_0 > 0$ *such that for all* $h \in (0, h_0)$

$$\mathcal{B}\Big(\lambda_{n,0} + \lambda_{n,1} h, c_0 h\Big) \cap \mathrm{sp}\left(\mathfrak{L}_h\right) = \{\lambda_n(h)\}.$$

Thanks to our theorem giving the splitting of the lowest eigenvalues, we have sharp asymptotic expansions of the eigenvalues. In particular, one knows that they become simple in the semiclassical limit and we get the approximation of the eigenfunctions by the WKB expansions.

When A_2 is not zero, it appears that the dimensional reduction is prevented by the oscillations of the eigenfunctions of the model operator $\mathcal{M}_{x,\xi}$. The problem already appears in the case $t \in \mathbb{R}$: we can gauge out A_2 at the price to replace A_1 by $A_1 + h\nabla_s\varphi(s,t)$ which is h dependent. As a consequence of our analysis, we can check that the spectrum associated with the potential $(A_1 + h\nabla_s\varphi, 0)$ is shifted by a factor $\mathcal{O}(h)$ compared to the one associated with $(A_1, 0)$. In dimension one for t, we can even prove with our method (and a change of gauge) that the phase Φ in the WKB expansion is (s,t)-dependent.

Scheme of the proof

The new Ansatz considered here is given by a partial WKB expansion with respect to the variable s. Under our analyticity assumptions, the effective eikonal equation is solved thanks to the classical stable manifold theorem and analytic extensions of the eigenpairs of the "model" operator $\mathcal{M}_{x,\xi}$. The corresponding effective transport equation is obtained as the Fredholm condition of an operator valued transport equation jointly with the Feynman-Hellmann formulas.

Flavor of the proof

We write

$$\mathfrak{L}_h^\natural = D_t^2 + (hD_s + A^\natural)^2, \qquad A^\natural(s,t) = \xi_0 + A_1(x_0 + s, t).$$

In order to lighten the notation, we introduce

$$\mathcal{M}^\natural_{x,\xi} = \mathcal{M}_{x+x_0,\xi+\xi_0}, \qquad u^\natural_{x,\xi} = u_{x+x_0,\xi+\xi_0}, \qquad \mu^\natural(x,\xi) = \mu(x+x_0,\xi+\xi_0).$$

We have:

$$\left(\mathcal{M}^\natural_{x,\xi}\right)^* = \mathcal{M}^\natural_{x,\bar{\xi}}, \qquad \forall x \in \mathbb{R}^m, \xi \in \mathbb{C}^m.$$

The assumption $A_2 = 0$ implies the fundamental property:

$$\overline{u^\natural_{x,\xi}} = u^\natural_{x,\bar{\xi}}\,.$$

We conjugate $\mathfrak{L}_h^\natural$ via a weight function $\Phi = \Phi(s)$ and define

$$\begin{aligned}\mathfrak{L}_\Phi^\natural &= \mathrm{e}^{\Phi(s)/h}\ \mathfrak{L}_h^\natural\ \mathrm{e}^{-\Phi(s)/h}\\ &= D_t^2 + (hD_s + i\nabla\Phi + A^\natural)^2\\ &= \mathfrak{L}_0^\natural + h\mathfrak{L}_1^\natural + h^2\mathfrak{L}_2^\natural,\end{aligned}$$

with

$$\begin{aligned}\mathfrak{L}_0^\natural &= D_t^2 + (i\nabla\Phi + A^\natural)^2 = \mathcal{M}^\natural_{s,i\nabla\Phi(s)},\\ \mathfrak{L}_1^\natural &= \frac{1}{2}\Big(D_s\cdot(\nabla_\xi\mathcal{M}^\natural)_{s,i\nabla\Phi(s)} + (\nabla_\xi\mathcal{M}^\natural)_{s,i\nabla\Phi(s)}\cdot D_s\Big),\\ \mathfrak{L}_2^\natural &= D_s^2\Phi.\end{aligned}$$

We now look for a formal solution in the form

$$\lambda \sim \sum_{j\geq 0} \lambda_j h^j, \qquad a \sim \sum_{j\geq 0} a_j h^j$$

such that $\mathfrak{L}^{\natural}_{\Phi} a = \lambda a$.

We have to find (λ_0, a_0) such that

$$\mathfrak{L}_0^{\natural} a_0 = \lambda_0 a_0 .$$

We must choose

$$\lambda_0 = \mu_0 .$$

Thus we have to find a_0 such that

(1) $$\mathfrak{L}_0^{\natural} a_0 = \mu_0 a_0 ,$$

that is to say

$$\mathcal{M}^{\natural}_{s, i\nabla\Phi(s)} a_0 = \mu_0 a_0 .$$

To solve (1), we choose a_0 in the form

$$a_0(s,t) = u^{\natural}_{s, i\nabla\Phi(s)}(t) b_0(s),$$

where b_0 has to be determined and Φ is a solution of the following eikonal equation (justified by our analyticity assumptions)

$$\mu^{\natural}(s, i\nabla_s \Phi) = \mu_0 .$$

Collecting the terms in h^1, we obtain the first transport equation

$$(\mathfrak{L}_0^\natural - \mu_0)a_1 = -(\mathfrak{L}_1^\natural - \lambda_1)a_0.$$

Pointwise in s, the Fredholm compatibility condition writes

$$(\lambda_1 - \mathfrak{L}_1^\natural)a_0 \in (\mathrm{Ker}({\mathfrak{L}_0^\natural}^* - \mu_0))^\perp.$$

We have $\mathrm{Ker}({\mathfrak{L}_0^\natural}^* - \mu_0) = \mathsf{span}(u^\natural_{s,-i\nabla\overline{\Phi}(s)})$, so that the compatibility condition is equivalent to

$$\lambda_1 \left\langle u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\mathrm{d}t)} = \left\langle \mathfrak{L}_1^\natural u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\Phi(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\mathrm{d}t)}, \ \forall s \in \mathbb{R}^m.$$

By using a Feynman-Hellmann formula, we are led to introduce

$$\mathsf{T} = \frac{1}{2}\left(\nabla_\xi \mu^\natural \cdot D_s + D_s \cdot \nabla_\xi \mu^\natural\right),$$

and we get the equation

$$\mathsf{T} b_0 = \lambda_1 b_0.$$

Then, λ_1 has to be chosen to solve the linearized transport equation at the singular point $s = 0$ and this condition is nothing but the belonging to the spectrum of the "harmonic oscillator" of symbol $\frac{1}{2}\mathrm{Hess}_{x_0,\xi_0}\mu$.

Inspired by Mikael and Søren...

Let us introduce a family of magnetic Laplacians in dimension two which is related to [HP10] and the more recent result by Fournais and Persson [FP13]. For $k \in \mathbb{N} \setminus \{0\}$, we consider the operator on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}s\,\mathrm{d}t)$:

$$\mathcal{L}_{\hbar}^{[k],\mathrm{gM}} = \hbar^2 D_{\mathsf{t}}^2 + \left(\hbar D_{\mathsf{s}} - \gamma(\mathsf{s})\frac{\mathsf{t}^{k+1}}{k+1}\right)^2,$$

where γ does not vanish.

Assumption (simple well)

The function γ admits a unique minimum $\gamma_0 > 0$ at $s_0 = 0$ which is non degenerate and $\lim_{s\to\pm\infty} \gamma(s) = +\infty$.

In order to stick to the previous analysis, we start by the following naive but fundamental rescaling

$$\mathsf{s} = s, \qquad \mathsf{t} = \hbar^{\frac{1}{k+2}} t.$$

The operator becomes

$$\hbar^{\frac{2k+2}{k+2}} \left(D_t^2 + \left(\hbar^{\frac{1}{k+2}} D_s - \gamma(s) \frac{t^{k+1}}{k+1} \right)^2 \right).$$

The investigation is then reduced to the one of

$$\mathfrak{L}_h^{[k]} = D_t^2 + \left(h D_s - \gamma(s) \frac{t^{k+1}}{k+1} \right)^2,$$

with $h = \hbar^{\frac{1}{k+2}}$.

Let us present three models for which the geometry is more intricate and for which our main theorem does not directly apply.

Vanishing or not vanishing

Assumption

The zero locus of $\mathbf{B}$ *is a smooth closed non empty curve* Γ*:*

$$\Gamma = \{\mathbf{B}(x) = 0\},$$

and $\mathbf{B}$ *vanishes exactly at the order* $k \geq 1$ *on* Γ*. Moreover we assume that the* k*-th normal derivative of* $\mathbf{B}$ *admits a non degenerate minimum on* Γ *at* x_0*.*

The following considerations can be extended to the case $k = 0$ which can be interpreted as follows. The curve Γ represents the boundary of an enclosed open set Ω carrying a magnetic Neumann condition (with a magnetic field which does not vanish on $\overline{\Omega}$).

Vanishing or not vanishing

If $k \geq 0$ is an integer, we let $\Omega_k = \mathbb{R}$ if $k \geq 1$ and $\Omega_k = \mathbb{R}_+$ if $k = 0$. By using the standard tubular coordinates near Γ, we are reduced to analyze the following operator, depending on the integer $k \geq 0$ and acting on $\mathrm{L}^2(\mathbb{R} \times \Omega_k, (1 - t\kappa(s))\,\mathrm{d}s\,\mathrm{d}t)$ and with Neumann condition on $t = 0$ if $k = 0$:

$$\mathcal{L}_{\hbar}^{\mathrm{vf},[k]} = (1 - t\kappa(s))^{-1}\hbar D_t(1 - t\kappa(s))\hbar D_t$$
$$+ (1 - t\kappa(s))^{-1}(\hbar D_s - A^{\mathrm{vf},[k]}(s,t))(1 - t\kappa(s))^{-1}(\hbar D_s - A^{\mathrm{vf},[k]}(s,t)),$$

with

$$A^{\mathrm{vf},[k]}(s,t) = \gamma(s)\frac{t^{k+1}}{k+1} + \tilde{\delta}(s)\frac{t^{k+2}}{k+2} + \mathcal{O}(t^{k+3}).$$

Vanishing or not vanishing

Assumption

The functions κ and B are smooth, γ is analytic and admits a positive and non degenerate minimum at $s=0$.

Let us perform the rescaling

$$h = \hbar^{\frac{1}{k+2}}, \qquad s = \sigma, \qquad t = h\tau.$$

We denote by $\mathfrak{L}_h^{\mathrm{vf},[k]}$ the rescaled operator divided by h^{2k+2}:

$$\mathfrak{L}_h^{\mathrm{vf},[k]} = (1-h\tau\kappa(\sigma))^{-1}D_\tau(1-h\tau\kappa(\sigma))D_\tau$$
$$+(1-h\tau\kappa(\sigma))^{-1}(hD_\sigma - {A_h}^{\mathrm{vf},[k]}(\sigma,\tau))(1-h\tau\kappa(\sigma))^{-1}(hD_\sigma - {A_h}^{\mathrm{vf},[k]}(\sigma,\tau)),$$

with

$${A_h}^{\mathrm{vf},[k]}(\sigma,\tau) = h^{-(k+1)}A^{\mathrm{vf},[k]}(\sigma,h\tau).$$

Theorem

There exist a function $\Phi = \Phi(\sigma)$ *defined in a neighborhood* $\mathcal{V}$ *of* $(0,0)$ *with* $\Re \mathrm{Hess}\, \Phi(0) > 0$ *and, for any* $n \geq 1$*, a sequence of real numbers* $(\lambda_{n,j}^{\mathrm{vf},[k]})_{j\geq 0}$ *such that*

$$\lambda_n^{\mathrm{vf},[k]}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j}^{\mathrm{vf},[k]} h^j,$$

in the sense of formal series. Besides there exists a formal series of smooth functions on $\mathcal{V}$

$$a_n^{\mathrm{vf},[k]}(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathrm{vf},[k]} h^j, \text{ with } a_{n,0}^{\mathrm{vf},[k]}(0,0) \neq 0 \text{ such that}$$

$$\left(\mathfrak{L}_h{}^{\mathrm{vf},[k]} - \lambda_n^{\mathrm{vf},[k]}(h)\right)\left(a_n(.;h)\mathrm{e}^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right)\mathrm{e}^{-\Phi/h}.$$

Theorem (continued)

We also have that $\lambda_{n,0}^{\mathrm{vf},[k]} = \gamma(0)^{\frac{2}{k+2}}\nu^{[k]}(\zeta_0^{[k]})$ *and that* $\lambda_{n,1}^{\mathrm{vf},[k]}$ *is the n-th eigenvalue of the operator*

$$\frac{1}{2}\mathrm{Hess}\ \mu^{[k]}(0,\zeta_0^{[k]})(\sigma, D_\sigma) + R^{\mathrm{vf},[k]}(0),$$

with

$$\begin{aligned} &R^{\mathrm{vf},[k]}(0) \\ &= 2\gamma(0)\left(\delta(0) + \frac{\kappa(0)\gamma(0)}{k+1}\right)\int_{\Omega_k} \frac{\tau^{2k+3}}{(k+1)(k+2)}\left(u_{0,i\Phi'(0)}^{\mathrm{vf},[k]}(\tau)\right)^2 \mathrm{d}\tau \\ &\qquad + \kappa(0)\int_{\Omega_k} \partial_\tau u_{0,i\Phi'(0)}^{\mathrm{vf},[k]}(\tau) u_{0,i\Phi'(0)}^{\mathrm{vf},[k]}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Theorem (continued)

The main term in the Ansatz is

$$a_{n,0}^{\mathrm{vf},[k]}(\sigma,\tau) = f_{n,0}^{\mathrm{vf},[k]}(\sigma) u_{\sigma,i\Phi'(\sigma)}^{\mathrm{vf},[k]}(\tau),$$

where $f_{n,0}^{\mathrm{vf},[k]}(\sigma)$ is the solution of the effective transport equation associated to the n-th eigenvalue of the effective "harmonic oscillator". Moreover, for all $n \geq 1$, there exist $h_0 > 0$, $c > 0$ such that for all $h \in (0, h_0)$, we have

$$\mathcal{B}(\lambda_{n,0}^{\mathrm{vf},[k]} + \lambda_{n,1}^{\mathrm{vf},[k]} h, ch) \cap \mathrm{sp}\left(\mathfrak{L}_h^{\mathrm{vf},[k]}\right) = \{\lambda_n^{\mathrm{vf},[k]}(h)\},$$

and $\lambda_n^{\mathrm{vf},[k]}(h)$ is a simple eigenvalue.

Along a varying edge

Our strategy can also deal with more singular situations in dimension three. Such a situation is described in [PR13] where the semiclassical analysis (simplicity of the eigenvalues) is done when the boundary of the domain contains a varying edge. It is possible to perform the WKB constructions for a simplified version of the operator introduced there. The operator, defined on $\mathrm{L}^2(\mathcal{W}_\alpha, \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$ and with Neumann conditions, is

$$\mathcal{L}_\hbar^{\mathrm{e}} = \hbar^2 D_t^2 + \hbar^2 D_z^2 + (\hbar D_s - t)^2,$$

where

$$\mathcal{W}_\alpha = \left\{(s,t,z) \in \mathbb{R}^3, |z| \leq \mathcal{T}(s)t\right\},$$

with $\mathcal{T}(s) = \tan\left(\frac{\alpha(s)}{2}\right)$ and where $\alpha : \mathbb{R} \to \mathbb{R}$ is an analytic function which represents the (varying) opening of the wedge $\mathcal{W}_\alpha$.
If the function $s \mapsto \alpha(s)$ admits a unique and non degenerate maximum α_0 at $s = 0$, then we can perform a WKB construction near $s = 0$.

Curvature induced magnetic bound states

A fundamental result of Helffer and Morame establishes that a smooth Neumann boundary can trap the lowest eigenfunctions near the points of maximal curvature. These considerations are generalized in [FH06, Theorem 1.1] where the complete asymptotic expansion of the eigenpairs is proved

$$\lambda^{c}_{n,\hbar} = \Theta_0 \hbar - C_1 \kappa_{\max} \hbar^{3/2} + (2n-1) C_1 \Theta_0^{1/4} \sqrt{\frac{3k_2}{2}} \hbar^{7/4} + o(\hbar^{7/4}),$$

where $k_2 = -\kappa''(0)$. Here, as in [FH06], we will consider the magnetic Neumann Laplacian on a smooth domain Ω such that the algebraic curvature κ satisfies the following assumption.

Assumption

The function κ is smooth and admits a unique and non degenerate maximum at 0.

Curvature induced magnetic bound states

Let us consider the following Neumann realization on $\mathrm{L}^2(\mathbb{R}^2_+, m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$, $\mathcal{L}^{\mathsf{c}}_{\hbar}$ expressed as

$$m(s,t)^{-1}\hbar D_t m(s,t)\hbar D_t + m(s,t)^{-1}\mathcal{P}_{\hbar} m(s,t)^{-1}\mathcal{P}_{\hbar},$$

where $m(s,t) = 1 - t\kappa(s)$ and $\mathcal{P}_{\hbar} = \hbar D_s + \zeta_0 \hbar^{\frac{1}{2}} - t + \kappa(s)\frac{t^2}{2}$.
Thanks to the rescaling

$$h = \hbar^{1/2}, \qquad t = h\tau, \qquad s = \sigma,$$

and after division by h^2 the operator $\mathcal{L}^{\mathsf{c}}_{\hbar}$ becomes $\mathfrak{L}^{\mathsf{c}}_{h}$ operating on the space $\mathrm{L}^2(m(\sigma, h\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau)$.

Theorem

There exist a function

$$\Phi = \Phi(\sigma) = \left(\frac{2C_1}{\nu''(\zeta_0)}\right)^{1/2} \left|\int_0^\sigma (\kappa(0) - \kappa(\varsigma))^{1/2}\, \mathrm{d}\varsigma\right|$$

defined in a neighborhood $\mathcal{V}$ *of* $(0,0)$ *such that* $\Re\Phi''(0) > 0$, *and a sequence of real numbers* $(\lambda^{\mathsf{c}}_{n,j})_{j\geq 0}$ *such that*

$$\lambda^{\mathsf{c}}_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda^{\mathsf{c}}_{n,j} h^{\frac{j}{2}}.$$

Besides there exists a formal series of smooth functions on $\mathcal{V}$,

$$a^{\mathsf{c}}_n \underset{h\to 0}{\sim} \sum_{j\geq 0} a^{\mathsf{c}}_{n,j} h^{\frac{j}{2}}$$

such that $$(\mathfrak{L}^{\mathsf{c}}_h - \lambda^{\mathsf{c}}_n(h))\left(a^{\mathsf{c}}_n \mathrm{e}^{-\Phi/h^{\frac{1}{2}}}\right) = \mathcal{O}\left(h^\infty\right)\mathrm{e}^{-\Phi/h^{\frac{1}{2}}}.$$

Theorem (continued)

We also have that $\lambda^{c}_{n,0} = \Theta_0$, $\lambda^{c}_{n,1} = 0$, $\lambda^{c}_{n,2} = -C_1\kappa_{\max}$ *and* $\lambda^{c}_{n,3} = (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$. *The main term in the Ansatz is in the form*

$$a^{c}_{n,0}(\sigma,\tau) = f^{c}_{n,0}(\sigma)u_{\zeta_0}(\tau).$$

Moreover, for all $n \geq 1$, *there exist* $h_0 > 0$, $c > 0$ *such that for all* $h \in (0, h_0)$, *we have*

$$\mathcal{B}\left(\lambda^{c}_{n,0} + \lambda^{c}_{n,2}h + \lambda^{c}_{n,3}h^{\frac{3}{2}}, ch^{\frac{3}{2}}\right) \cap \mathrm{sp}\left(\mathfrak{L}^{c}_{h}\right) = \{\lambda^{c}_{n}(h)\},$$

and $\lambda^{c}_{n}(h)$ *is a simple eigenvalue.*

In particular, this theorem proves that there are no odd powers of $\hbar^{\frac{1}{8}}$ in the expansion of the eigenvalues (see [FH06, Theorem 1.1]).

Zebra camels

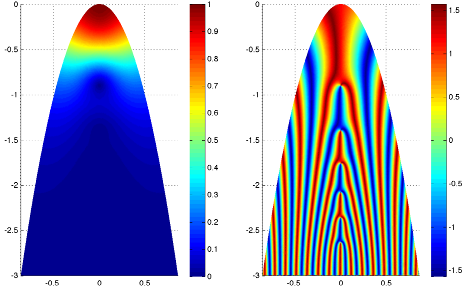

Figure: Modulus and phase of the first eigenfuntion, $h = \frac{1}{20}$.

Zebra camels with two bumps

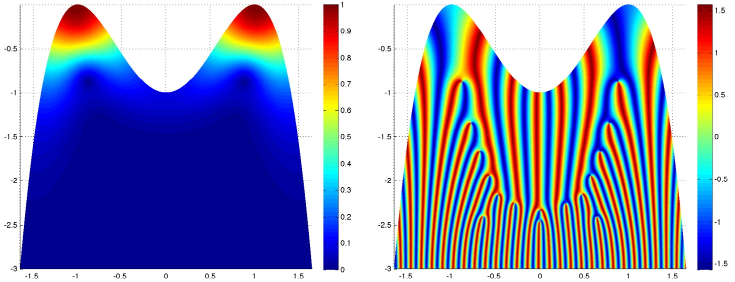

Figure: Modulus and phase of the first eigenfunction, $h = \frac{1}{20}$.

Zebra camels with two bumps

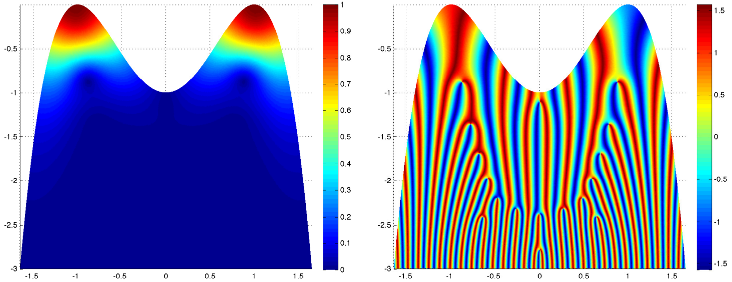

Figure: Modulus and phase of the second eigenfunction, $h = \frac{1}{20}$.

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6. Nevertheless, our WKB constructions can be used to get a formal estimate of the splitting.

Before I compose a piece, I walk around it several times, accompanied by myself.

Erik Satie