Invariants of disordered topological insulators

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What is a topological insulator?

- d-dimensional disordered system of independent Fermions with basic symmetries and having non-trivial topology in the bulk
- Gap or Anderson localization regime at Fermi level
- Basic symmetries are a combination of

TRS, PHS, SLS = time reversal, particle hole, sublattice symmetry

- Topology of bulk = of Bloch bundles of periodic approximants:

winding numbers, Chern numbers, $\mathbb{Z}_2$-invariants, higher invariants

- Delocalized edge modes with non-trivial topology
- Bulk-edge correspondence, stable w.r.t. disorder
- Toy models: tight-binding, Dirac-like operators
- Wider notions include interactions, bosons, spins, photonic crys.

Examples of topological insulators in $d = 2$:

- Integer quantum Hall systems (no symmetries at all)
- Quantum spin Hall systems (Kane-Mele 2005, odd TRS)

dissipationless spin polarized edge currents, spin-charge separation

- Dirty superconductors (Bogoliubov-de Gennes BdG models):

Thermal quantum Hall effect (even PHS)

Majorana modes at Landau-Ginzburg vortices (even PHS)

Spin quantum Hall effect (SU(2)-invariant, odd PHS)

- Example in $d = 1$: Kitaev chain with Majorana modes
- Example in $d = 3$: chiral unitary systems

Menu for the talk

- Some standard background on Fredholm operators
- Review of quantum Hall systems (focus on topology)
- Classification of $d = 2$ topological insulators
- Needed: Fredholm operators with symmetries
- More on the physics of $d = 2$ systems: QSH and BdG

Fredholm operators and Noether indices

Definition $T \in \mathcal{B}(\mathcal{H})$ bounded Fredholm operator on Hilbert space

$\iff T\mathcal{H}$ closed, $\dim(\mathrm{Ker}(T)) < \infty$, $\dim(\mathrm{Ker}(T^*)) < \infty$

Then: $\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ran}(T))$ Noether index

Theorem $\mathrm{Ind}(T)$ compactly stable homotopy invariant

Noether Index Theorem $f \in C(\mathbb{S}^1)$ invertible, Π Hardy on $L^2(\mathbb{S}^1)$

$\Longrightarrow \mathrm{Wind}(f) = \int f^{-1} df = -\mathrm{Ind}(\Pi f \Pi)$

Atiyah-Singer index theorems in differential topology

Alain Connes non-commutative geometry and topology

Applications in physics Anomalies in QFT, Defects, etc.

Solid state physics robust labelling of different phases

Problem determine Fredholm operator in concrete situation

Review of quantum Hall system (no symmetries)

Toy model: disordered Harper Hamiltonian on Hilbert space $\ell^2(\mathbb{Z}^2)$

$$H \;=\; U_1 + U_1^* + U_2 + U_2^* + \lambda_{\rm dis} V$$

$U_1 = e^{i\varphi X_2} S_1$ and $U_2 = S_2$ with magnetic flux φ and $S_{1,2}$ shifts

random potential $V = \sum_{n\in\mathbb{Z}^2} V_n |n\rangle\langle n|$ with i.i.d. $V_n \in \mathbb{R}$

Fermi projection $P = \chi(H \leq \mu)$ with μ in And. localization regime

Theorem (Connes, Bellissard, Kunz, Avron, Seiler, Simon ...)

$$PFP \quad \text{Fredholm operator} \qquad , \qquad F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$$

Index equal to Chern number

$$\mathrm{Ind}(PFP) = \mathrm{Ch}(P) = 2\pi i\, \mathbb{E}\, \langle 0| P\, [[X_1, P], [X_2, P]] |0\rangle$$

$$= \int \frac{d^2 k}{2\pi i}\, \mathrm{Tr}_q (P\, [\partial_1 P, \partial_2 P])$$

Physical consequences

Theorem

(Thouless et.al. 1982, Avron, Seiler, Simon 1983-1994, Kunz 1987, Bellissard, van Elst, S-B 1994, ...)

Kubo formula for zero temperature Hall conductivity $\sigma_H(\mu)$

$$\sigma_H(\mu) \;=\; \frac{e^2}{h}\,\mathrm{Ch}(P)$$

and $\mu \in \Delta \mapsto \sigma_H(\mu)$ constant if Anderson localization in $\Delta \subset \mathbb{R}$

Theorem

(Rammal, Bellissard 1985, Resta 2010, S-B, Teufel 2013)

$M(\mu) = \partial_B p(T = 0, \mu)$ orbital magnetization at zero temperature

$$\partial_\mu M(\mu) \;=\; \mathrm{Ch}(P) \qquad \mu \in \Delta$$

Link to spectral flow (Laughlin argument 1981)

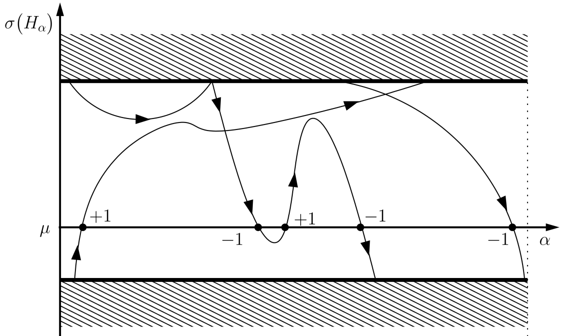
Folk involves adiabatics; for Landau see Avron, Pnuelli (1992)

Theorem (Macris 2002, Nittis, S-B 2014)

Hamiltonian $H(\alpha)$ with extra flux $\alpha \in [0,1]$ through 1 cell of $\mathbb{Z}^2$

$H(\alpha) - H$ compact, so only discrete spectrum close to μ in gap

$$\mathrm{Ch}(P) \;=\; \mathrm{Spectral\ Flow}\Big(\alpha \in [0,1] \mapsto H(\alpha) \;\text{ through }\; \mu\Big)$$



Bulk-edge correspondence

Edge currents in periodic systems: Halperin 1982, Hatsugai 1993

Theorem (S-B, Kellendonk, Richter 2000, 2002)

$\mu \in \Delta$ gap of H and $\widehat{H}$ restriction to half-space $\ell^2(\mathbb{Z} \times \mathbb{N})$

With $g : \mathbb{R} \to [0,1]$ increasing from 0 to 1 in Δ

$$\widehat{\mathcal{T}}(g'(\widehat{H})\,\widehat{J}_1) \;=\; \mathrm{Ch}(P)$$

where $\widehat{J}_1 = i[X_1, \widehat{H}] = \nabla_1 \widehat{H}$ current operator and

$$\widehat{\mathcal{T}}(\widehat{A}) \;=\; \sum_{x_2 \geq 0} \mathbb{E}\,\langle 0, x_2|\,\widehat{A}\,|0, x_2\rangle \qquad \text{tracial state on edge ops}$$

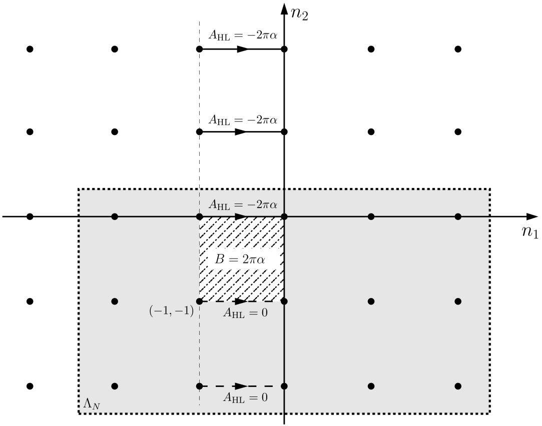
Moreover, link to winding number of $\widehat{V} = \exp(2\pi i\, g(\widehat{H}))$

$$\mathrm{Ch}(P) \;=\; i\,\widehat{\mathcal{T}}(\widehat{V}^*\nabla_1\widehat{V})$$

without gap condition: Elgart, Graf, Schenker 2005

Macris' argument for bulk-edge correspondence

$$\mathrm{Ch}(P) \;=\; \mathrm{Ind}(PFP) \;=\; -\int_0^1 d\alpha\;\mathrm{Tr}\big(g'(\widetilde{H}^N_\alpha)\,\partial_\alpha \widetilde{H}^N_\alpha\big)$$



Tight-binding toy models in dimension $d = 2$

Hilbert space $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$

Fiber $\mathbb{C}^L = \mathbb{C}^{2s+1} \otimes \mathbb{C}^r$ with spin s and r internal degrees

e.g. $\mathbb{C}^r = \mathbb{C}^2_{\rm ph} \otimes \mathbb{C}^2_{\rm sl}$ particle-hole space and sublattice space

Typical Hamiltonian

$$H \;=\; \sum_{i=1}^{4} \left(W_i^* U_i + W_i U_i^*\right) \,+\, W_0 \,+\, \lambda_{\rm dis}\, V$$

U_1 and U_2 magnetic translations as above

next nearest neighbor $U_3 = U_1^* U_2$ and $U_4 = U_1 U_2$

W_i matrices $L \times L$ (e.g. for spin orbit coupling, pair creation)

Matrix potential $V = V^* = \sum_{n \in \mathbb{Z}^2} V_n |n\rangle\langle n|$ random (i.i.d.)

Fredholm operator PFP where $P = \chi(H \leq \mu)$ and $F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$

Implementing symmetries

$$\begin{array}{ll} \text{SLS (Chiral)} : & K_{\rm sl}^* \, H \, K_{\rm sl} \;=\; -H \\ \text{TRS} : & I_{\rm s}^* \, \overline{H} \, I_{\rm s} \;=\; H \\ \text{PHS} : & K_{\rm ph}^* \, \overline{H} \, K_{\rm ph} \;=\; -H \end{array}$$

$K_{\rm sl}$, $I_{\rm s}$ and $K_{\rm ph}$ real unitaries on fibers $\mathbb{C}^2_{\rm sl}$, $\mathbb{C}^{2s+1}$, $\mathbb{C}^2_{\rm ph}$ with

$$K_{\rm sl}^2 \;=\; \pm\mathbf{1} \qquad I_{\rm s}^2 \;=\; \pm\mathbf{1} \qquad K_{\rm ph}^2 \;=\; \pm\mathbf{1}$$

Sign of $K_{\rm sl}$ irrelevant, but TRS and PHS can be even or odd

Example: $I_{\rm s} = e^{i\pi s^y}$ even/odd = integer/half-integer spin

Note: TRS + PHS $\implies$ SLS with $K_{\rm sl} = I_{\rm s} K_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes, 2 complex and 8 real

Classification of $d = 2$ topological insulators

Schnyder, Ryu, Furusaki, Ludwig 2008, reordering Kitaev 2008

Nittis, S-B 2014: classification with $T = PFP$ (strong invariants)

CAZ	TRS	PHS	SLS	System	symmetry	Ind	Topology
A	0	0	0	QHE	none	$\mathbb{Z}$	$K^0(\mathbb{R}^2)$
AIII	0	0	1		$K_{\rm sl}^* T K_{\rm sl} = T^c$	0	$K^1(\mathbb{R}^2)$
D	0	$+1$	0	TQH	none	$\mathbb{Z}$	$KR^2(\mathbb{R}^2)$
DIII	-1	$+1$	1	SCS	two	$\mathbb{Z}_2$	$KR^3(\mathbb{R}^2)$
AII	-1	0	0	QSH	$I_{\rm s}^* T^t I_{\rm s} = T$	$\mathbb{Z}_2$	$KR^4(\mathbb{R}^2)$
CII	-1	-1	1		two	0	$KR^5(\mathbb{R}^2)$
C	0	-1	0	SQH	Ker(T) quat.	$2\,\mathbb{Z}$	$KR^6(\mathbb{R}^2)$
CI	$+1$	-1	1		two	0	$KR^7(\mathbb{R}^2)$
AI	$+1$	0	0		$I_{\rm s}^* T^t I_{\rm s} = T$	0	$KR^0(\mathbb{R}^2)$
BDI	$+1$	$+1$	1		two	0	$KR^1(\mathbb{R}^2)$

Milnor: $KR^n(\mathbb{R}^2) = \pi_{n-3}(O)$ where O stable orthogonal group

$\mathbb{Z}_2$ indices of odd symmetric Fredholm operators

$I = I_s$ real unitary on Hilbert space $\mathcal{H}$ with real structure, $I^2 = -\mathbf{1}$

Definition T odd symmetric $\iff I^* T^t I = T$ with $T^t = (\overline{T})^*$

Theorem (S-B 2013)

$\mathbb{F}_2(\mathcal{H}) = \{$odd symmetric Fredholm operators$\}$ has 2 connected components labeled by the compactly stable homotopy invariant:

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \ \in \ \mathbb{Z}_2$$

Class AII (QSH): H odd TRS $\iff I^* \overline{H} I = H \iff I^* H^t I = H$

So: H odd symmetric $\implies H^n$ odd sym. $\implies f(H)$ odd sym.

Fermi projection P odd sym. and PFP odd sym. Fredholm

$$\mathrm{Ind}_2(PFP) \in \mathbb{Z}_2 \text{ well-defined} \qquad , \qquad F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$$

Also for Fermi level in region of dynamically localized states!

Proofs for $\mathbb{Z}_2$ indices (S-B 2013)

Proposition Even degeneracies for odd symmetric matrices.

Proof: odd symmetry $I^* T^t I = T \implies (IT)^t = -IT$

$\implies \det(T - z\,\mathbf{1}) = \det(IT - z\,I) = \mathrm{Pf}(IT - z\,I)^2$ $\square$

Similar to Kramers' degeneracy, but no invariance under $\psi \mapsto I\overline{\psi}$

Proposition K compact odd symmetric

$\implies \mathbf{1} + K$ even degeneracies and $\mathrm{Ind}_2(\mathbf{1} + K) = 0$

This is a weak form of compact stability, namely at $T = \mathbf{1}$

Theorem (Siegel) T odd symmetric $\iff T = I^* A^t I A$

Proof of connectedness:

$\mathrm{Ind}_2(T) = 0 \implies T$ invertible (mod $\mathcal{K}$) $\implies A$ invertible

$s \in [0,1] \mapsto A_s$ homotopy to $\mathbf{1}$

$\implies s \in [0,1] \mapsto T_s = I^*(A_s)^t I A_s$ path to $\mathbf{1}$ in odd symmetrics

Quantum spin Hall system (odd TRS, Class AII)

Disordered Kane-Mele model on hexagon lattice and with $s = \frac{1}{2}$

$$H \;=\; \Delta_{\rm hexagon} + H_{\rm SO} + H_{\rm Ra} + \lambda_{\rm dis} V$$

Pseudo-gap at Dirac point opens non-trivially due to

$$H_{\rm SO} \;=\; i\,\lambda_{\rm SO} \sum_{i=1,2,3} \left(S_i^{\rm nn} - (S_i^{\rm nn})^*\right) s^z$$

No s^z-conservation due to Rashba term $H_{\rm Ra}$, but odd TRS

Non-trivial topology:

Kane-Mele (2005): $\mathbb{Z}_2$ invariant for periodic system from Pfaffians

Haldane et al. (2005): spin Chern numbers for s^z invariant systems

Prodan (2009): spin Chern number from $P_{\rm s} = \chi(|Ps^zP - \frac{1}{2}| < \frac{1}{2})$

$$\mathrm{SCh}(P) = \mathrm{Ch}(P_{\rm s}) \;\in\; \mathbb{Z}$$

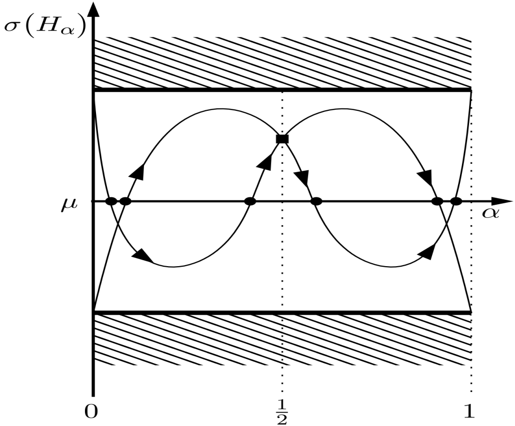
Periodic systems: Avila, S-B, Villegas 2013, Graf, Porta 2013

$\mathbb{Z}_2$ invariant and spin-charge separation

Theorem $\mathrm{Ind}_2(PFP)$ phase label for odd TRS

Theorem (Nittis, S-B, 2014)

$$\mathrm{Ind}_2(PFP) = 1 \quad \Longrightarrow \quad H(\alpha = \tfrac{1}{2}) \text{ has TRS + Kramers pair in gap}$$



Spin filtered helical edge channels for QSH

Theorem (S-B 2013)

$\mathrm{Ind}_2(PFP) = 1 \implies$ spin Chern numbers $\mathrm{SCh}(P) \neq 0$

Remark Non-trivial topology $\mathrm{SCh}(P)$ persists TRS breaking!

Theorem (S-B 2012)

If $\mathrm{SCh}(P) \neq 0$, dissipationless spin filtered edge currents are stable w.r.t. perturbations by magnetic field and disorder:

$$\widehat{\mathcal{T}}\big(g'(\widehat{H})\,\tfrac{1}{2}\{\widehat{J}_1, s^z\}\big) \;=\; \mathrm{SCh}(P) \;+\; \text{correct.}$$

Resumé: $\mathrm{Ind}_2(PFP) = 1 \Longrightarrow$ no Anderson loc. for edge states

Rice group: Du, Knez, et al since 2011 in InAs/GaSb Bilayers

Four-terminal conductance plateaux stable w.r.t. magnetic field

Here spin Chern number is relevant and not $\mathbb{Z}_2$ invariant!

Resumé

- Topological insulators have non-trivial Bloch theory
- Invariants persist under weak disorder
- Edge states are not exposed to Anderson localization
- Physical effects have to be studied case by case