Functions of self-adjoint operators in ideals of compact operators: applications to the entanglement entropy

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Introduction

We discuss two groups of estimates for functions of self-adjoint operators.

1. Let $A = A^*$ in a Hilbert space $\mathcal{H}$, and let P be an orthogonal projection. We study

$$W(A, P; f) = Pf(PAP)P - Pf(A)P$$

with suitable functions $f : \mathbb{R} \to \mathbb{C}$.
Aim: to obtain estimates for the (quasi-)norms of $W(A, P; f)$ in various ideals of compact operators.
In particular, in the Schatten-von Neumann ideals $\mathbf{S}_p$ with the (quasi-)norm

$$\|A\|_p = \left[\sum_{k=1}^{\infty} s_k(A)^p\right]^{\frac{1}{p}}, \ 0 < p < \infty.$$

Here $s_k(A) = \lambda_k(|A|), k = 1, 2, \dots,$ are singular values (s-values) of the operator A. Here $|A| = (A^*A)^{1/2}$. If $p \geq 1$(resp. $p < 1$), then the above formula defines a norm (resp. quasi-norm).

Laptev-Safarov 1996: if $f \in \mathsf{W}^{2,\infty}$ and $PA \in \mathbf{S}_2$, then

$$|\operatorname{tr}\big(Pf(PAP)P - Pf(A)P\big)| \le \frac{1}{2}\|f''\|_{\mathsf{L}^\infty}\|(I-P)AP\|_2^2.$$

Our aim is to derive similar bounds with other (quasi-)norms and functions with singularities. Typical example: $f(t) = (t-a)^\gamma, \gamma > 0$, a fixed.

2. Let $A = A^*$ on $\mathcal{H}_2$, $B = B^*$ on $\mathcal{H}_1$, and let $J : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Interested in the bounds for

$$f(A)J - Jf(B)$$

in (quasi-)normed ideals of compact operators.

Note that for $J = P$, $B = PAP$ we have

$$Pf(A)P - Pf(PAP)P = P\big(f(A)J - Jf(B)\big)P.$$

Quasi-normed ideals

Let $\mathbf{S} \subset \mathbf{S}_\infty$ be a two-sided ideal of compact operators. The functional $\| \cdot \|_{\mathbf{S}}$ is a symmetric quasi-norm if

1. $\|A\|_{\mathbf{S}} > 0, A \neq 0$;
2. $\|zA\|_{\mathbf{S}} = |z| \|A\|_{\mathbf{S}}$;
3. $\|A_1 + A_2\|_{\mathbf{S}} \le \varkappa(\|A_1\|_{\mathbf{S}} + \|A_2\|_{\mathbf{S}}),\ \varkappa \ge 1$;
4. $\|XAY\|_{\mathbf{S}} \le \|X\| \|Y\| \|A\|_{\mathbf{S}}$;
5. For every one-dimensional operator A: $\|A\|_{\mathbf{S}} = \|A\|$.

Useful observations:

1. $\|A\|_{\mathbf{S}}$ depends only on the s-values $s_k(A)$, $k = 1, 2, \dots$!
2. Any quasi-norm has an equivalent q-norm with a suitable $q \in (0, 1]$, i.e.
$$\|A_1 + A_2\|_{\mathbf{S}}^q \le \|A_1\|_{\mathbf{S}}^q + \|A_2\|_{\mathbf{S}}^q.$$
In fact, q is found from the equation $\varkappa = 2^{q^{-1}-1}$.
For $\mathbf{S} = \mathbf{S}_p, p \in (0, 1)$: Yu. Rotfeld 1967.

Survey

1. $J = I$, $\mathbf{S} = \mathbf{S}_1$:

$$\|f(A) - f(B)\|_1 \le C\|A - B\|_1, C = C(f). \tag{1}$$

   V. Peller 1985,90: $f \in B^1_{\infty 1}$ implies (**??**), and (**??**) implies $f \in B^1_{11}$ locally.
   Let $f_\beta(t) = |t|^\beta \zeta(t), \zeta \in \mathsf{C}_0^\infty(\mathbb{R})$. Then $f_\beta \in B^1_{\infty 1}$ if $\beta > 1$ and $f_\beta \in B^1_{11}$ for all $\beta > 0$.
   The property $f \in B^1_{11}$ is sharp, R. Frank–A. Pushnitski, 2013.
2. D. Potapov–F. Sukochev 2011: If $f \in \mathrm{Lip}(\mathbb{R})$ and $p > 1$, then

$$\|f(A) - f(B)\|_p \le C(f)\|A - B\|_p. \tag{2}$$

   where $C(f) = C_1\|f\|_{\mathrm{Lip}}$.
3. The case $p \in (0, 1)$ was studied by V. Peller 1987 for unitary A, B: $f \in B^{1/p}_{\infty p}$ implies (**??**), and (**??**) implies $f \in B^{1/p}_{pp}$.

4. Let $\mathbf{S}$ be a normed ideal with the *majorization property*, i.e. if $A \in \mathbf{S}$, $B \in \mathbf{S}_\infty$, and

$$\sum_{k=1}^{n} s_k(B) \le \sum_{k=1}^{n} s_k(A), \ \forall n = 1, 2, \dots,$$

then $B \in \mathbf{S}$ and $\|B\|_{\mathbf{S}} \le \|A\|_{\mathbf{S}}$.
M. Birman–L. Koplienko–M. Solomyak 1975: if $A \ge 0, B \ge 0$, then for any $\gamma \in (0, 1)$:

$$\|A^\gamma - B^\gamma\|_{\mathbf{S}} \le \||A - B|^\gamma\|_{\mathbf{S}}.$$

For $\mathbf{S} = \mathbf{S}_1$ reproved by E.H. Lieb–H. Siedentop–J. P. Solovej, 1997.

5. A. Aleksandrov–V. Peller 2010: if $f \in \Lambda_\gamma = B^\gamma_{\infty\infty}$, $\gamma \in (0,1)$ (Zygmund class), then

$$\| |f(A)J - Jf(B)|^{\frac{1}{\gamma}} \|_{\mathbf{S}} \le C(f) \|J\|^{\frac{1-\gamma}{\gamma}} \|AJ - JB\|_{\mathbf{S}},$$

under the condition that the *Boyd index* $\beta(\mathbf{S})$ is < 1!
For example, $\beta(\mathbf{S}_p) = p^{-1}$.

Define the operator $[A]_d = \oplus_{k=1}^d A$ on $\oplus_{k=1}^d \mathcal{H}$.
If $A \in \mathbf{S}$ then $[A]_d \in \mathbf{S}$. Denote by $\beta_d(\mathbf{S})$ the quasi-norm of the transformer $T \to [T]_d$ in $\mathbf{S}$. Then the *Boyd index* of $\mathbf{S}$ is defined to be

$$\beta(\mathbf{S}) = \lim_{d\to\infty} \frac{\log \beta_d(\mathbf{S})}{\log d}.$$

Results

Assumptions.

Operators: $A = A^*$ on $\mathcal{H}_2$, $B = B^*$ on $\mathcal{H}_1$, $J : \mathcal{H}_1 \to \mathcal{H}_2$. Define the form

$$V[f, g] = (Jf, Ag) - (JBf, g),\ \ f \in D(B) \subset \mathcal{H}_1, g \in D(A) \subset \mathcal{H}_2,$$

and suppose that

$$|V[f, g]| \leq C\|f\|\|g\|,$$

so that V defines a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Functions: $f \in \mathsf{C}^n(\mathbb{R} \setminus \{0\})$, and

$$|f^{(k)}(t)| \leq C|t|^{\beta - k}\chi_1(t),\ \ k = 0, 1, \dots, n,$$

for some $\beta > 0$. Here χ_1 is the indicator of the interval $(-1, 1)$. Denote

$$\| f \|_n = \max_{0 \leq k \leq n} \max_t |f^{(k)}(t)||t|^{k-\beta}.$$

Theorem

Let $\mathbf{S}$ *be a* q*-normed ideal with* $(n-\gamma)^{-1} < q \le 1$*, where* $\gamma \in (0,1]$*,* $\gamma < \beta$*. Suppose that* V *is such that* $|V|^\gamma \in \mathbf{S}$*. Then*

$$\|f(A)J - Jf(B)\|_{\mathbf{S}} \le C \|\!| f \|\!|_n \|J\|^{1-\gamma} \||V|^\gamma\|_{\mathbf{S}}.$$

Assume that $\mathbf{S} = \mathbf{S}_p$, $\beta > 1$, $J = I$. Is it possible to use one of Peller's earlier results to get the bound

$$\|f(A) - f(B)\|_p \le C\|V\|_p?$$

No! Indeed, consider $f(t) = |t|^\beta \zeta(t), \zeta \in C_0^\infty(\mathbb{R})$, $\zeta(t) = 1$, $|t| < 1$. Then $f \in B^\nu_{p,q}$, $0 < p \le \infty, 0 < q < \infty$ iff $\nu < \beta + p^{-1}$. For sufficiently small p, we have $f \notin B^{1/p}_{\infty,p}$. On the other hand, $f \in B^{1/p}_{p,p}$.

Remark. If $|g^{(k)}(t)| \le C|t|^{\beta-k}\chi_R(t)$, then

$$\|g(A)J - Jg(B)\|_{\mathbf{S}} \le CR^{\beta-\gamma} \| g \|_n \|J\|^{1-\gamma} \| |V|^\gamma \|_{\mathbf{S}}.$$

Apply the Theorem to $f(t) = R^{-\beta} g(Rt)$ with $A' = R^{-1}A, B' = R^{-1}B$.

For $J = P$ and $B = PAP$ we have $V = (I - P)AP$.

Theorem

$$\|g(A)P - Pg(PAP)\|_{\mathbf{S}} \le CR^{\beta-\gamma} \| g \|_n \| |(I-P)AP|^\gamma \|_{\mathbf{S}}.$$

Quasi-analytic extension

Let $f \in \mathsf{C}^1(\mathbb{R} \setminus \{0\}) \cap \mathsf{C}_0(\mathbb{R})$, and let $\tilde{f} \in \mathsf{C}^1(\mathbb{R}^2 \setminus \{0\}) \cap \mathsf{C}_0(\mathbb{R}^2)$ be a function such that $\tilde{f}(x, 0) = f(x)$, and

$$|y|^{-1} \frac{\partial}{\partial \overline{z}} \tilde{f}(x, y) \in \mathsf{C}(\mathbb{R}^2 \setminus \{0\}) \cap \mathsf{L}^1(\mathbb{R}^2).$$

Then $\tilde{f}$ is called a quasi-analytic extension of f.

Proposition (Dyn'kin, Hörmander, Helffer-Sjöstrand)

Let A be a self-adjoint operator, and let $\tilde{f}$ be a quasi-analytic extension of f. Then

$$f(A) = \frac{1}{\pi} \iint \frac{\partial}{\partial \overline{z}} \tilde{f}(x, y)(A - x - iy)^{-1} dxdy.$$

For the function f:

$$\tilde{f}(x,y) = \left[\sum_{l=0}^{n-1} f^{(l)}(x)\frac{(iy)^l}{l!}\right]\sigma(x,y),\ \sigma(x,y) = \zeta\left(\frac{y}{x}\right),$$

with $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$, $\zeta(t) = 0$, $|t| \ge 1$; $\zeta(t) = 1$, $|t| \le 1/2$. Then

$$\left|\frac{\partial}{\partial \overline{z}}\tilde{f}(x,y)\right| \le C_n \| f \|_n |x|^{\beta-n}|y|^{n-1}\chi_1(x),$$

and $\partial_{\overline{z}}\tilde{f}$ is supported in the sector $|y| \le |x|$.

Application

Let $\Lambda, \Omega \in \mathbb{R}^d$, be domains, and let $\chi_\Lambda, \chi_\Omega$ be their characteristic functions. We are interested in the operators of the form

$$T_\alpha(a) = \chi_{\alpha\Lambda} a(D) \chi_\Omega(D) \chi_{\alpha\Lambda}, \ D = -i\nabla.$$

$a = \overline{a} \in \mathsf{C}_0^\infty(\mathbb{R}^d)$, $\alpha >> 1$.
We may interpret T_α as a (multi-dimensional) Wiener-Hopf operator with a discontinuous symbol.
Straightforward: if $a \equiv 1$, $|\Lambda|, |\Omega| < \infty$, then $T_\alpha \in \mathfrak{S}_1$ and

$$\|T_\alpha\|_{\mathfrak{S}_1} = \frac{1}{(2\pi)^d} \int \chi_{\alpha\Lambda}(\mathbf{x}) d\mathbf{x} \ \int \chi_\Omega(\boldsymbol{\xi}) d\boldsymbol{\xi} = \frac{\alpha^d}{(2\pi)^d} \ |\Lambda| \ |\Omega|.$$

Natural asymptotic problem:
Asymptotics of $\operatorname{tr} g(T_\alpha)$, $\alpha \to \infty$, with a suitable function $g : g(0) = 0$.

Two-term asymptotics

Conjecture (H. Widom, 1982):

$$\operatorname{tr} g(T_\alpha) = \alpha^d W_0 + \alpha^{d-1}\log\alpha\ W_1 + o(\alpha^{d-1}\log\alpha),\ \alpha\to\infty,$$

$$W_0 = W_0(g(a)) = \left(\frac{1}{2\pi}\right)^d \int_\Omega\int_\Lambda g(a(\boldsymbol\xi))d\mathbf{x}d\boldsymbol\xi,$$

$$W_1 = \left(\frac{1}{2\pi}\right)^{d-1}\frac{1}{4\pi^2}\int_{\partial\Omega}\int_{\partial\Lambda}|\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\boldsymbol\xi}|U(a(\boldsymbol\xi);g)dS_{\mathbf{x}}dS_{\boldsymbol\xi}.$$

$$U(a;g) = \int_0^1 \frac{g(ta)-tg(a)}{t(1-t)}dt.$$

Main result

Theorem (A.S. 2009, 2013)

Let $d \geq 2$. Suppose that Λ, Ω are bounded and

- *Both $\partial\Lambda$ and $\partial\Omega$ are Lipschitz.*
- *$\partial\Lambda$ is piece-wise C^1, and $\partial\Omega$ is piece-wise C^3.*

Then the Conjecture holds for any $g \in \mathsf{C}^\infty(\mathbb{R})$ s.t. $g(0) = 0$.

Theorem (A.S. 2014)

Let Λ, Ω be as above. Suppose that $g \in \mathsf{C}^2(\mathbb{R} \setminus \{0\}) \cap \mathsf{C}(\mathbb{R})$ and $|g^{(k)}(t)| \leq C|t|^{\beta-k}\chi_1(t)$, $k = 0, 1, 2$ with some $\beta > 0$. Then the Conjecture holds.

Use the previous results with $P = \chi_{\alpha\Lambda}$ and $A = a(D)\chi_\Omega(D)$.

Crucial ingredient:

Theorem

Let $\alpha \geq 2$. *Then for any* $q \in (0, 1]$*:*

$$\|(I - \chi_{\alpha\Lambda})a(D)\chi_\Omega(D)\chi_{\alpha\Lambda}\|_{\mathbf{S}_q}^q \leq C_q \alpha^{d-1} \log \alpha.$$

Take a function $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ such that $\zeta(t) = 1, |t| < 1$, and $\zeta(t) = 0, |t| \geq 2$. Denote $\zeta_\delta(t) = \zeta(t\delta^{-1})$. Split g: $g = g_1 + g_2$ where $g_1 = g\zeta_\delta$, $g_2 = g(1 - \zeta_\delta) \in \mathsf{C}^\infty$. Then for any $0 < q < \beta, q \leq 1$:

$$\|g_1(T_\alpha) - \chi_{\alpha\Lambda} g_1(a(D)\chi_\Omega(D))\chi_{\alpha\Lambda}\|_1 \leq C\delta^{\beta - q}\alpha^{d-1} \log \alpha.$$

For g_2 one can use the already known result. Notice that $|W_1(g_1)| \leq C(\delta^\beta + \delta)$. The required result follows.

Example: the entanglement entropy

Set $a = 1$, so $T_\alpha = \chi_{\alpha\Lambda}\chi_\Omega(D)\chi_{\alpha\Lambda}$.
"Free Fermions at zero temperature":

$$\Omega = \{\boldsymbol{\xi} : E(\boldsymbol{\xi}) < E_F\}.$$

Entanglement entropy (von Neumann entropy) between Fermions inside $\alpha\Lambda$ and outside $\alpha\Lambda$, $\alpha \to \infty$:

$$\mathcal{E}_\alpha = \operatorname{tr} h(T_\alpha),\ \ h(t) = -t\log t - (1-t)\log(1-t), t \in [0,1].$$

We see that

$$\mathcal{E}_\alpha = \frac{\alpha^{d-1}\log\alpha}{12(2\pi)^{d-1}}\int_{\partial\Omega}\int_{\partial\Lambda}|\mathbf{n}_\mathbf{x}\cdot\mathbf{n}_{\boldsymbol{\xi}}|d\mathbf{x}d\boldsymbol{\xi} + o(\alpha^{d-1}\log\alpha).$$

D. Gioev–I. Klich, 2006: the formula.
H. Leschke–W. Spitzer–A. S., 2013/14: rigorous proof.

The Rényi entropy: $\quad \eta_\beta(t) = \dfrac{1}{1-\beta}\log\big(t^\beta + (1-t)^\beta\big), \beta > 0.$