# The classical entropy of quantum states[1]

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[1] *Joint work with Elliott Lieb*

# Abstract of Talk

- To quantify the inherent uncertainty of quantum states Wehrl ('79) suggested a definition of their **classical entropy** based on the coherent state transform.
- He conjectured that this **classical entropy** is **minimized** by states that also minimize the Heisenberg uncertainty inequality, i.e., **Gaussian coherent states**.
- Lieb ('78) proved this conjecture and conjectured that the same holds when Euclidean **Glauber coherent states** are replaced by SU(2) **Bloch coherent states**.
- This conjecture was settled last year in joint work with Lieb. Recently we simplified the proof and generalized it to $SU(N)$ for general $N$. I will present this here.
- In proving the conjecture we study the quantum channels known as **Universal Quantum Cloning Machines** and determine their **minimal output entropy**. Thanks to K. Bradler for pointing this out.

# Outline of Talk

1. Coherent states and quantization
2. States of minimal classical entropy
3. $SU(N)$-coherent states
4. Classical $SU(N)$ entropy inequality
5. Generalization to Quantum Channels
6. Formulation in terms of majorization
7. Using bosonic 2nd quantization
8. A normal ordering formula
9. The classical limit (if time permits)

# Quantization

- **Classical phase space:** $\mathcal{M} = \mathbb{R}^{2n}$ position and momentum $(q, p)$.
- **Quantum description:** Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$.
- **Quantization:** Function $A$ on $\mathcal{M}$ to operator $Op(A)$ on $\mathcal{H}$.
- **Pure states**[2]**:** Described by normalized $\psi \in L^2(\mathbb{R}^n)$ gives "distribution" on phase space $\Phi_\psi$ such that

$$\langle \psi, Op(A)\psi \rangle = (2\pi)^{-n} \iint \Phi_\psi(q,p) A(q,p) dq dp$$

- **Weyl quantization** leads to $\Phi_\psi(q,p)$ **Wigner distribution**, which is **not necessarily positive**.
- Better to use **Wick** or **coherent state** quantization

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[2] The state is really represented by the 1-dim projection $|\psi\rangle\langle\psi|$. More general non-pure states represented by density matrices (operators): $0 \leq \rho$, $\operatorname{Tr}\rho = 1$.

# Coherent state quantization

**Coherent states**, i.e., states of minimal Heisenberg uncertainty

$$f_{q,p}(x) = \pi^{-n/4} \exp(-(x-q)^2/2 + ipx) \in L^2(\mathbb{R}^n)$$

satisfy $(x + \nabla) f_{q,p} = (q + ip) f_{q,p}$.
They define **quantization map**

$$Op(A) = (2\pi)^{-n} \iint A(q,p) |f_{q,p}\rangle\langle f_{q,p}| dq dp.$$

leads to **lower** or **covariant symbol** or **Husimi $Q$-function**

$$\Phi_\psi(q,p) = |\langle f_{q,p}|\psi\rangle|^2.$$

Then $0 \leq \Phi_\psi(q,p) \leq 1$ and $(2\pi)^{-n} \iint \Phi_\psi(q,p) dq dp = 1$.
**Wehrl classical entropy:**

$$S^{\mathrm{cl}}(\psi) = (2\pi)^{-n} \iint -\Phi_\psi(q,p) \log\left(\Phi_\psi(q,p)\right) dq dp.$$

# States of minimal entropy

**Theorem (Lieb '78, Conjectured by Wehrl)**

*States of minimal entropy are states of minimal Heisenberg uncertainty, i.e., for all $\psi$ and all $q, p$*

$$S^{\mathrm{cl}}(\psi) \geq S^{\mathrm{cl}}(f_{q,p}).$$

Proof based on sharp **Young** and **Hausdorff-Young** inequalities[3].
Carlen '91 proved **"uniqueness"** of minimizers.
In fact, $-t\log(t)$ may be replaced by any **concave function**:

**Theorem (Lieb-Solovej '12)**

*For all continuous concave* $f : [0,1] \to \mathbb{R}$, $f(0) = 0$

$$\iint f\left(\Phi_\psi(q',p')\right) dq'dp' \geq \iint f\left(\Phi_{f_{q,p}}(q',p')\right) dq'dp'$$

[3]Note that both Y and HY inequalities are optimized by Gaussians

# $SU(N)$ coherent states

- Consider **the Hilbert space** $\mathcal{H}_M = \bigotimes_{\mathrm{SYM}}^M \mathbb{C}^N$, i.e., the space of $M$ **Bosons** with $N$ degrees of freedom.
- $SU(N)$ acts **irreducibly** on $\mathcal{H}_M$ (not all irr. repr. unless $N=2$).
- **Special states** on $\mathcal{H}_M$, **coherent vectors, highest weight vectors, pure condensates:** $\otimes^M u$, $u \in \mathbb{C}^N$.
- The state $|\otimes^M u\rangle\langle\otimes^M u|$ depends only on the unit vector $u \in \mathbb{C}^N$ **modulo a phase**, i.e., really $u \in \mathbb{CP}^{N-1}$
- $\mathbb{CP}^{N-1}$ is a **classical phase space** and $\mathcal{H}_M$ is a quantization. **Quantization map:**

$$Op(A) = \dim \mathcal{H}_M \int_{\mathbb{CP}^{N-1}} A(u)|\otimes^M u\rangle\langle\otimes^M u|du, \quad Op(1) = I$$

  $du$ is $SU(N)$ **invariant (Liouville) measure** on $\mathbb{CP}^{N-1}$.
- **Husimi** function for general state $\rho$ on $\mathcal{H}_M$

$$\Phi^\infty(\rho)(u) = \langle\otimes^M u|\rho|\otimes^M u\rangle,$$

# Classical $SU(N)$ entropy inequality

**Theorem (Classical "entropy" inequality, Lieb-Solovej '13)**

*For all integers $M, N$, all concave $f : [0,1] \to \mathbb{R}$, all states $\rho$ on $\mathcal{H}_M$, and all $v \in \mathbb{CP}^{N-1}$*

$$\int_{\mathbb{CP}^{N-1}} f\left(\Phi^\infty(\rho)(u)\right) du \geq \int_{\mathbb{CP}^{N-1}} f\left(\Phi^\infty(| \otimes^M v\rangle\langle\otimes^M v|)(u)\right) du$$

- For $N = 2$, i.e., $SU(2)$, $\mathbb{CP}^{N-1} = \mathbb{S}^2$ is the **Bloch sphere**. In this case and for the **entropy function** $f(t) = -t\log(t)$ the result was conjectured by Lieb 1978. A proof of this case is to appear soon and is on the archive.
- Special cases of $M$ for $N = 2$ and the entropy function had been considered by Schupp '99, Scutaru '02
- For $N > 2$ the compact manifold $\mathbb{CP}^{N-1}$ is **not** a sphere.

# Generalization to Quantum channels

$\Phi^\infty$ is a map from a quantum state to a classical prob. distribution. We generalize to **completely positive trace preserving** maps, i.e., **quantum channels** $\Phi^k$ from operators on $\mathcal{H}_M$ to operators on $\mathcal{H}_{M+k}$,

$$\Phi^k(\rho) = C_{M,N.k} P_{\rm sym}(\rho \otimes I_{\otimes^k \mathbb{C}^N}) P_{\rm sym}.$$

The value of the **normalization constant** $C_{M,N.k}$ is not important here. The channels $\Phi^k$ are known as **universal quantum cloners**. We determine their **minimal output entropy**.

**Theorem (Lieb-Solovej '13)**

*For all $M$, all $k$, all concave $f : [0,1] \to \mathbb{R}$, all states $\rho$ on $\mathcal{H}_M$, and all $v \in \mathbb{CP}^{N-1}$*

$$\mathrm{Tr}_{\mathcal{H}_{M+k}} f\left(\Phi^k(\rho)\right) \geq \mathrm{Tr}_{\mathcal{H}_{M+k}} f\left(\Phi^k(|\otimes^M v\rangle\langle\otimes^M v|)\right)$$

# Formulation in terms of majorization

The **classical entropy inequality** follows from the **classical limit:**

$$\lim_{k\to\infty} \frac{1}{\dim \mathcal{H}_{M+k}} \mathrm{Tr}_{\mathcal{H}_{M+k}} f\left(\frac{\dim \mathcal{H}_{M+k}}{\dim \mathcal{H}_M} \Phi^k(\rho)\right) = \int_{\mathbb{CP}^{N-1}} f\left(\Phi^\infty(\rho)(u)\right) du.$$

Alternatively to using traces of concave functions the previous theorem may be equivalently (**Karamata's Theorem**) rephrased as

**Theorem**

*For all states $\rho$ on $\mathcal{H}_M$ and all $v \in \mathbb{CP}^{N-1}$ the ordered eigenvalues of $\Phi^k(|\otimes^M v\rangle\langle\otimes^M v|)$ majorizes the ordered eigenvalues of $\Phi^k(\rho)$.*

**Def.** $a_1 \geq a_2 \geq \cdots \geq a_J$ **majorizes** $b_1 \geq b_2 \geq \cdots \geq b_J$ if

$$\sum_{j=1}^m a_j \geq \sum_{j=1}^m b_j, \; m \leq J-1, \quad \text{and} \quad \sum_{j=1}^J a_j = \sum_{j=1}^J b_j.$$

# Using bosonic 2nd quantization

We introduce the **Bosonic annihilation** operators $a_i$, $i = 1, \ldots, N$ (indexing a basis $e_i$ of $\mathbb{C}^N$) and their adjoints the **creation** operators $a_i^*$:

$$a_i^* : \bigoplus_{M=0}^{\infty} \mathcal{H}_M \to \bigoplus_{M=0}^{\infty} \mathcal{H}_M, \qquad a_i^*(\mathcal{H}_M) \subseteq \mathcal{H}_{M+1}$$

$$a_i^* \phi = \sqrt{M+1} P_{\mathrm{sym}}(e_i \otimes \phi) \text{ for } \phi \in \mathcal{H}_M.$$

Then in **Kraus form**

$$\Phi^k(\rho) = C'_{M,N,k} \sum_{i_1,\ldots,i_k} a_{i_1}^* \cdots a_{i_k}^* \rho a_{i_k} \cdots a_{i_1}$$

**Two observations:**

- Ordered eigenvalue sums are convex: may assume $\rho = |\psi\rangle\langle\psi|$.
- The non-zero eigenvalues of $\Phi^k(|\psi\rangle\langle\psi|)$ equal the non-zero eigenvalues (counting multiplicities) of the matrix

$$C'_{M,N,k} \langle \psi | a_{i_k} \cdots a_{i_1} a_{j_1}^* \cdots a_{j_k}^* | \psi \rangle.$$

# A normal ordering formula

The matrix (the outcome of the **transpose channel** to $\Phi^k$)

$$\Gamma_{i_1,\ldots,i_k;j_1,\ldots j_k} = \langle\psi|a_{i_k}\cdots a_{i_1}a^*_{j_1}\cdots a^*_{j_k}|\psi\rangle.$$

represents an operator $\Gamma$ on $\mathcal{H}_k$. It is the **anti-normal ordering** of the matrix elements of the **reduced $k$-particle density matrix**

$$(\gamma_\psi)_{i_1,\ldots,i_k;j_1,\ldots j_k} = \langle\psi|a^*_{j_1}\cdots a^*_{j_k}a_{i_k}\cdots a_{i_1}|\psi\rangle.$$

In fact, **normal ordering** gives (See also Chiribella '10)

$$\Gamma = \sum_{\ell=0}^{k} C_\ell \Phi^\ell(\gamma_\psi^{(k-\ell)})$$

for coefficients $C_\ell > 0$. The **majorization theorem** follows by **induction on $k$:** Induction start: $\Phi^0 = \mathrm{Id}$. Induction step:

$$\Phi^\ell(\gamma^{(k-\ell)}_{\otimes^M v}) = c_{M,k,\ell}\Phi^\ell(|\otimes^{k-\ell} v\rangle\langle\otimes^{k-\ell} v|), \quad (c_{M,k,\ell} = \mathrm{Tr}\,\gamma_\psi^{(k-\ell)})$$

majorizes $\Phi^\ell(\gamma_\psi^{(k-\ell)})$ for all $\ell < k$, but $\ell = k$ obvious.

# The classical limit (only one sided inequality)

Will show a version of the **Berezin-Lieb inequality**: For $f$ concave

$$\frac{1}{\dim \mathcal{H}_{M+k}} \mathrm{Tr}_{\mathcal{H}_{M+k}} f\left(\frac{\dim \mathcal{H}_{M+k}}{\dim \mathcal{H}_M} \Phi^k(\rho)\right) \leq \int_{\mathbb{CP}^{N-1}} f\left(\Phi^\infty(\rho)(u)\right) du$$

If $\rho = | \otimes^M v\rangle\langle \otimes^M v|$ right side explicitly limit $k \to \infty$ of left side:
**That is all we need!**
**Jensen's inequality** implies **Berezin-Lieb-inequality**:

$$\begin{aligned}
&\frac{1}{\dim \mathcal{H}_{M+k}} \mathrm{Tr}_{\mathcal{H}_{M+k}} f\left(\frac{\dim \mathcal{H}_{M+k}}{\dim \mathcal{H}_M} \Phi^k(\rho)\right) \\
&\qquad = \int_{\mathbb{CP}^{N-1}} \left\langle \otimes^{M+k} u \left| f\left(\frac{\dim \mathcal{H}_{M+k}}{\dim \mathcal{H}_M} \Phi^k(\rho)\right)\right| \otimes^{M+k} u \right\rangle du \\
&\qquad \leq \int_{\mathbb{CP}^{N-1}} f\left(\frac{\dim \mathcal{H}_{M+k}}{\dim \mathcal{H}_M} \left\langle \otimes^{M+k} u \left| \Phi^k(\rho) \right| \otimes^{M+k} u \right\rangle\right) du \\
&\qquad = \int_{\mathbb{CP}^{N-1}} f\left(\langle \otimes^M u | \rho | \otimes^M u\rangle\right) du = \int_{\mathbb{CP}^{N-1}} f\left(\Phi^\infty(\rho)(u)\right) du.
\end{aligned}$$