Peierls substitution for magnetic Bloch bands

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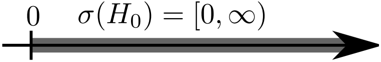
based on joint work with **Silvia Freund**.
(arXiv:1312.5931)

1. Introduction: Band spectra

- $H_0 = -\frac{1}{2}\Delta_x$ on $L^2(\mathbb{R}^d_x)$



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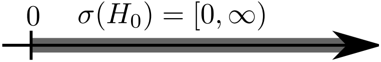
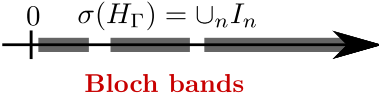


- $H_\Gamma = -\frac{1}{2}\Delta_x + V_\Gamma(x)$



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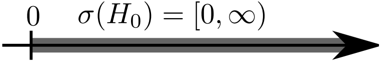

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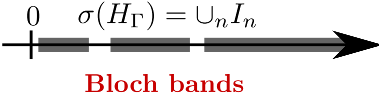

  with $V_\Gamma(x + \gamma) = V_\Gamma(x)$ for all $x \in \mathbb{R}^d$, $\gamma \in \Gamma \sim \mathbb{Z}^d$

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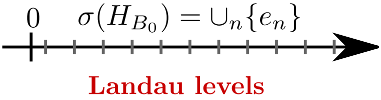
  with $\mathrm{d}A_0 = B_0 = \mathrm{const.}$



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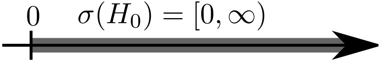

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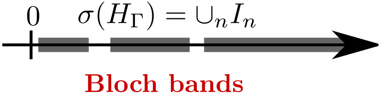

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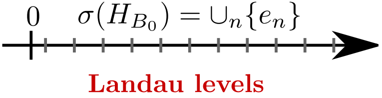
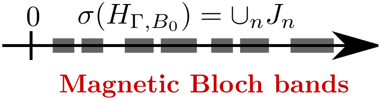


- $H_{\Gamma,B_0} = \frac{1}{2}(-\mathrm{i}\nabla_x + A_0(x))^2 + V_\Gamma(x)$



  with Γ and B_0 commensurable

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1. Introduction: Peierls substitution

What happens when we add to these operators a non-periodic potential W and a non-linear vector potential A corresponding to "small" fields?

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- Fourier transformation turns $\widetilde{H}_0 = \frac{1}{2}\big(-\mathrm{i}\nabla_x + A(x)\big)^2 + W(x)$ into the pseudo-differential operator

$$\widetilde{H}_0 \sim \frac{1}{2}\big(k + A(\mathrm{i}\nabla_k)\big)^2 + W(\mathrm{i}\nabla_k)\,.$$

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  The restriction of $\widetilde{H}_\Gamma = \frac{1}{2}\big(-\mathrm{i}\nabla_x + A(x)\big)^2 + V_\Gamma(x) + W(x)$ to one of the subspaces $L^2(\mathbb{T}^d)$ under Bloch-Floquet transformation should be close to

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1. Introduction: Some mathematical literature

- Buslaev '87
- Guillot, Ralston, Trubowitz '88
- Bellisard '88, '89
- Helffer, Sjöstrand '88, '89, '90
- Nenciu '89, '91
- Gérard, Martinez, Sjöstrand '91
- Panati, Spohn, T. '03
- Dimassi, Guillot, Ralston '04
- De Nittis, Panati '10
- De Nittis, Lein '11
- Cornean, Nenciu '14

2. Setup and scaling

Let $d = 2$, $B_0 \in \mathbb{R}$, $\mathcal{B}_0 := \begin{pmatrix} 0 & -B_0 \\ B_0 & 0 \end{pmatrix}$, $A_0(x) := \frac{1}{2}\mathcal{B}_0 x$

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and

$$\tilde{\Gamma} := \{a\tilde{\gamma}_1 + b\tilde{\gamma}_2 \in \mathbb{R}^2 \mid a, b \in \mathbb{Z}\}$$

for some basis $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ of $\mathbb{R}^2$.

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Let $V_{\tilde{\Gamma}} : \mathbb{R}^2 \to \mathbb{R}$ be periodic with respect to $\tilde{\Gamma}$,

$$V_{\tilde{\Gamma}}(x + \gamma) = V_{\tilde{\Gamma}}(x) \quad \text{for all } \gamma \in \tilde{\Gamma},\ x \in \mathbb{R}^2,$$

and relatively bounded with respect to $(-\mathrm{i}\nabla_x + A_0(x))^2$ with relative bound smaller than one.

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Then

$$H_{\mathrm{MB}} := \tfrac{1}{2}(-\mathrm{i}\nabla_x + A_0(x))^2 + V_{\tilde{\Gamma}}(x)$$

is self-adjoint on the magnetic Sobolev space $H^2_{A_0}(\mathbb{R}^2)$.

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Let $W \in C_b^\infty(\mathbb{R}^2, \mathbb{R})$ and $A \in C_b^\infty(\mathbb{R}^2, \mathbb{R}^2)$, where we choose a gauge for A such that $A(x) \cdot \tilde{\gamma}_2 = 0$ for all $x \in \mathbb{R}^2$.

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Then for $\varepsilon \in (0, \varepsilon_0]$ the magnetic Bloch Hamiltonian perturbed by slowly varying external fields is

$$H^{\varepsilon} := \tfrac{1}{2}\big(-\mathrm{i}\nabla_x + A_0(x) + A(\varepsilon x)\big)^2 + V_{\tilde{\Gamma}}(x) + W(\varepsilon x)\,,$$

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Note that all of the following works similarly for slowly perturbed tight binding models.

3. Magnetic Bloch Floquet transformation

Define the magnetic translation of functions on $\mathbb{R}^2$ by $\widetilde{\gamma}_j$ as

$$(\widetilde{T}_j\psi)(x) := \mathrm{e}^{\frac{\mathrm{i}}{2}\langle x, \mathcal{B}_0\widetilde{\gamma}_j\rangle}\psi(x - \widetilde{\gamma}_j)\,.$$

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$$\widetilde{T}_j^{-1}\,(-\mathrm{i}\nabla - A_0)\,\widetilde{T}_j = (-\mathrm{i}\nabla - A_0)\,, \quad \widetilde{T}_j^{-1}\,V_{\widetilde{\Gamma}}\,\widetilde{T}_j = V_{\widetilde{\Gamma}}$$

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Because of

$$\widetilde{T}_1\,\widetilde{T}_2 = \mathrm{e}^{\mathrm{i}\langle\widetilde{\gamma}_2, \mathcal{B}_0\widetilde{\gamma}_1\rangle}\,\widetilde{T}_2\,\widetilde{T}_1\,,$$

we only obtain a unitary representation of $\widetilde{\Gamma}$ if $\langle\widetilde{\gamma}_2, \mathcal{B}_0\widetilde{\gamma}_1\rangle \in 2\pi\mathbb{Z}$.

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Here $\langle\widetilde{\gamma}_2, \mathcal{B}_0\widetilde{\gamma}_1\rangle = B_0|M|$ is the magnetic flux through the unit cell M of the lattice $\widetilde{\Gamma}$ with volume $|M| = \widetilde{\gamma}_1 \wedge \widetilde{\gamma}_2$.

3. Magnetic Bloch Floquet transformation

Let the flux of B_0 per unit cell satisfy $\langle \widetilde{\gamma}_2, \mathcal{B}_0 \widetilde{\gamma}_1 \rangle = 2\pi \frac{p}{q} \in 2\pi\mathbb{Q}$.

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$$T : \Gamma \to \mathcal{L}(L^2(\mathbb{R}^2)), \quad \gamma = n_1\gamma_1 + n_2\gamma_2 \;\mapsto\; T_\gamma := T_1^{n_1} T_2^{n_2}$$

is a unitary representation of Γ on $L^2(\mathbb{R}^2)$ satisfying

$$T_\gamma^{-1} H_{\mathrm{MB}} T_\gamma = H_{\mathrm{MB}}$$

for all $\gamma \in \Gamma$.

3. Magnetic Bloch Floquet transformation

For $\psi \in C_0^\infty(\mathbb{R}^2)$ the magnetic Bloch-Floquet transformation is defined by

$$(\mathcal{U}_{\mathrm{BF}}\psi)(k, y) := \mathrm{e}^{-\mathrm{i}y\cdot k} \sum_{\gamma\in\Gamma} \mathrm{e}^{\mathrm{i}\gamma\cdot k}(T_\gamma\psi)(y) \, .$$

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$\hat{\psi} := \mathcal{U}_{\mathrm{BF}}\psi$ satisfies as a function on $\mathbb{R}^2_k \times \mathbb{R}^2_y$

$$T_\gamma\hat{\psi}(k,\cdot) = \hat{\psi}(k,\cdot) \quad \text{for all} \;\; k\in\mathbb{R}^2\,,\ \gamma\in\Gamma\,,$$

and

$$\hat{\psi}(k-\gamma^*,y) = \underbrace{\mathrm{e}^{\mathrm{i}\gamma^*\cdot y}}_{=:\tau(\gamma^*)}\,\hat{\psi}(k,y) \quad \text{for all} \;\; k,y\in\mathbb{R}^2\,,\ \gamma^*\in\Gamma^*$$

where Γ^* is the dual lattice to Γ.

3. Magnetic Bloch Floquet transformation

Introducing

$$\mathcal{H}_{\mathrm{f}} := \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^2_y) \,\middle|\, T_\gamma f = f \quad \text{for all} \quad \gamma \in \Gamma \right\} ,$$

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and

$$\mathcal{H}_\tau := \{ g \in L^2_{\mathrm{loc}}(\mathbb{R}^2_k, \mathcal{H}_{\mathrm{f}}) \,|\, g(k - \gamma^*) = \tau(\gamma^*) g(k) \quad \text{for all} \quad \gamma^* \in \Gamma^* \} ,$$

equipped with the inner product $\langle f, g \rangle_{\mathcal{H}_\tau} = \int_{M^*} \langle f(k), g(k) \rangle_{\mathcal{H}_{\mathrm{f}}} \mathrm{d}k$, the magnetic Bloch-Floquet transformation is a unitary map

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The Bloch-Floquet transform $\hat{H}_{\mathrm{MB}} := \mathcal{U}_{\mathrm{BF}} \, H_{\mathrm{MB}} \, \mathcal{U}^*_{\mathrm{BF}}$ of the unperturbed Hamiltonian H_{MB} acts on $\psi \in \mathcal{H}_\tau$ as

$$(\hat{H}_{\mathrm{MB}} \psi)(k) = H_{\mathrm{f}}(k) \psi(k) ,$$

where

$$H_{\mathrm{f}}(k) := \tfrac{1}{2} \big(-\mathrm{i}\nabla_y - A_0(y) + k \big)^2 + V_\Gamma(y) .$$

3. Magnetic Bloch bands

$H_{\mathrm{f}}(k)$ has discrete spectrum with eigenvalues $E_n(k)$ of finite multiplicity that accumulate at infinity. Let

$$E_1(k) \leq E_2(k) \leq \dots$$

be the eigenvalues repeated according to their multiplicity. In the following, $k \mapsto E_n(k)$ will be called the nth band function or just the nth Bloch band.

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Since $H_{\mathrm{f}}(k)$ is τ-equivariant, i.e.

$$H_{\mathrm{f}}(k - \gamma^*) = \tau(\gamma^*)\, H_{\mathrm{f}}(k)\, \tau(\gamma^*)^{-1}\,,$$

and $\tau(\gamma^*)$ is unitary, the Bloch bands $E_n(k)$ are Γ^*-periodic functions.

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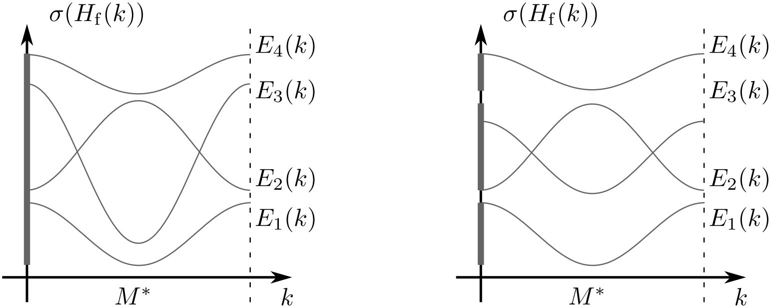
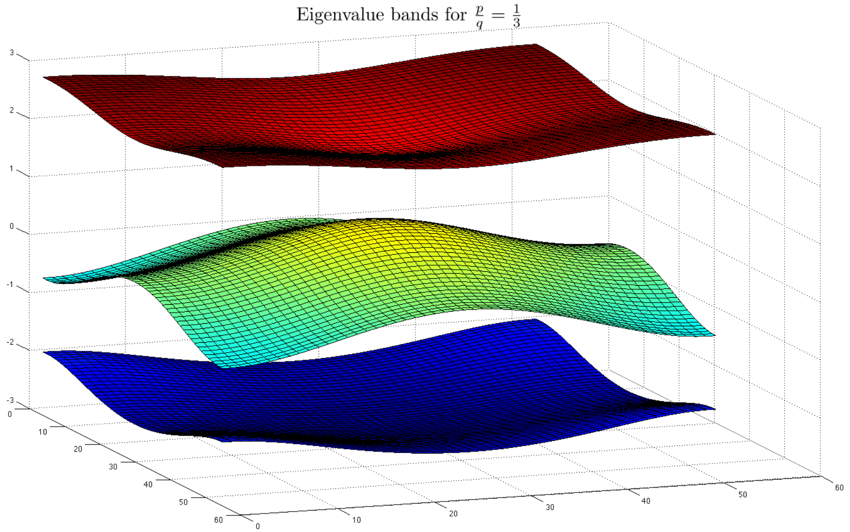


3. Magnetic Bloch bands: Hofstadter Hamiltonian



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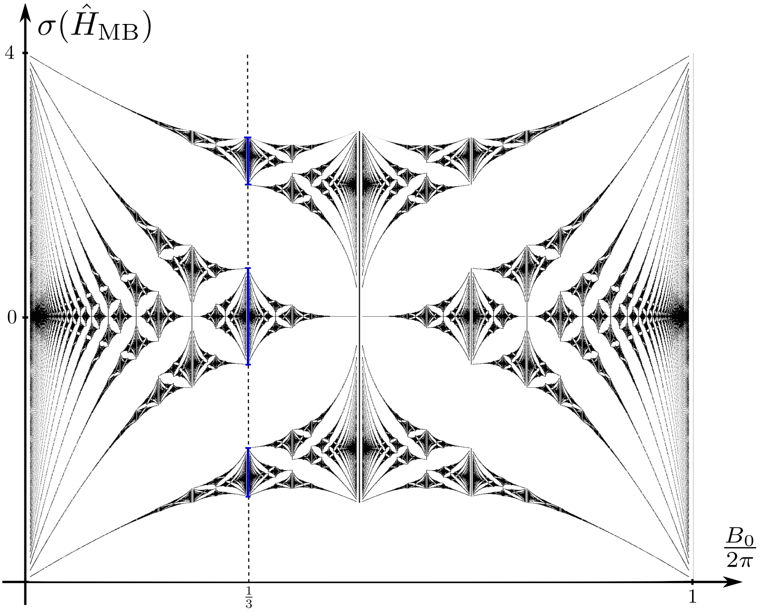

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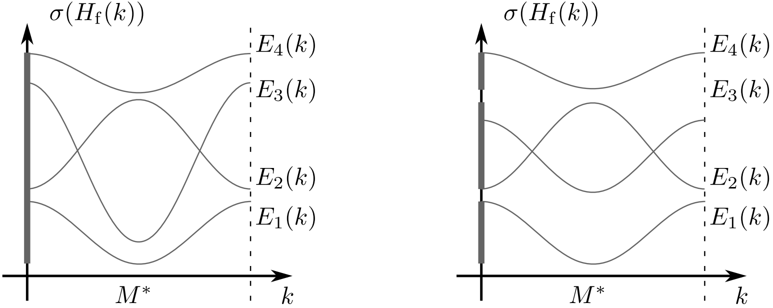

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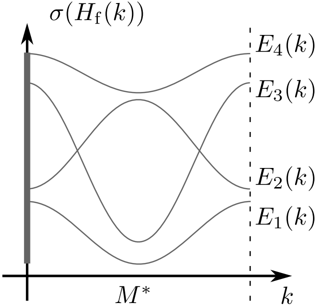

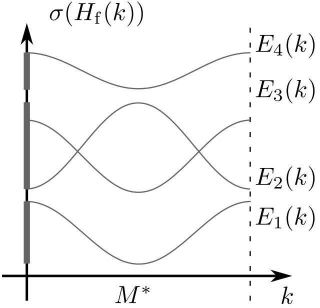

Let $P_n(k)$ be the spectral projection of $H_{\mathrm{f}}(k)$ corresponding to the eigenvalue $E_n(k)$.
Then on the subspace

$$P_n\mathcal{H}_\tau := \{\psi \in \mathcal{H}_\tau \mid \psi(k) \in P_n(k)\mathcal{H}_{\mathrm{f}}\}$$

we have

$$(\hat{H}_{\mathrm{MB}}\psi)(k) = H_{\mathrm{f}}(k)\psi(k) = E_n(k)\psi(k)\,.$$

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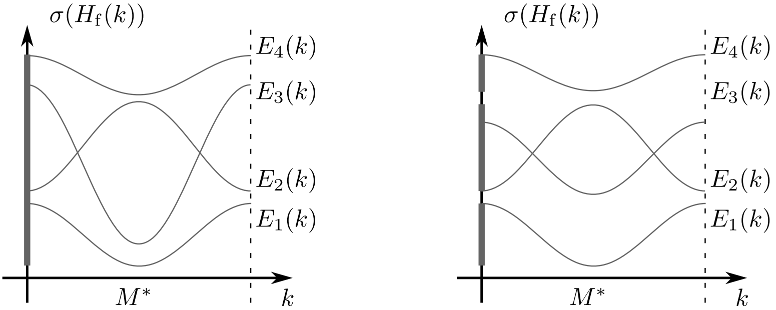

A family $\{E_n(k)\}_{n\in I}$ with $I = [I_-, I_+] \cap \mathbb{N}$ is called isolated, if

$$\inf_{k\in M^*} \operatorname{dist}\left(\cup_{n\in I}\{E_n(k)\}, \cup_{m\notin I}\{E_m(k)\}\right) > 0\,.$$

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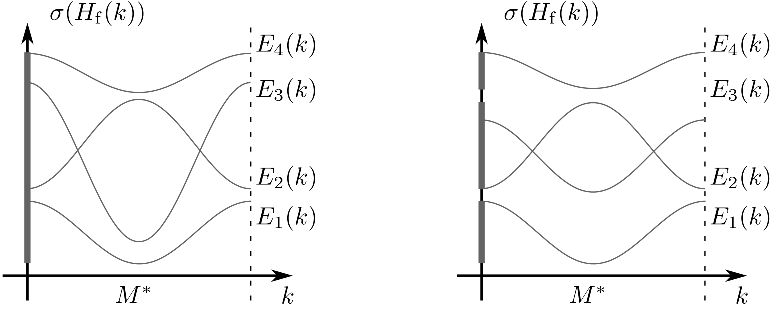

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We say that $\{E_n(k)\}_{n\in I}$ is strictly isolated, if for

$$\sigma_I := \overline{\cup_{n\in I} \cup_{k\in M^*} \{E_n(k)\}}$$

we have that
$$\inf_{m\notin I, k\in M^*} \operatorname{dist}(E_m(k), \sigma_I) > 0\,.$$

4. Results

Theorem

Let $\{E_n(k)\}_{n \in I}$ be an isolated family of Bloch bands. Then there exists an orthogonal projection $\Pi_I^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$ such that $H^\varepsilon \Pi_I^\varepsilon$ is a bounded operator and

$$\|[H^\varepsilon, \Pi_I^\varepsilon]\| = \mathcal{O}(\varepsilon^\infty)\,.$$

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Moreover, Π_I^ε is close to a pseudodifferential operator $\mathrm{Op}^\tau(\pi)$,

$$\|\Pi_I^\varepsilon - \mathrm{Op}^\tau(\pi)\| = \mathcal{O}(\varepsilon^\infty)\,, \qquad (*)$$

with principal symbol $\pi_0(k,r) = P_I(k - A(r))$.

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If $\{E_n(k)\}_{n\in I}$ is strictly isolated and if the gaps remain open for $\varepsilon \in (0, \varepsilon_0]$, then $(*)$ holds for Π_I^ε being the corresponding spectral projection of H^ε.

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The construction is well known and based on methods developed by *Helffer and Sjöstrand* in '89 that were applied in similar ways by *Martinez, Nenciu and Sordoni* in '03.

4. Results

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To prove "Peierls substitution", we need to show that $\Pi_n^\varepsilon H^\varepsilon \Pi_n^\varepsilon$ is unitarily equivalent to an operator of the form

$$H_n^{\mathrm{eff}} = E_n\big(k + A(\mathrm{i}\varepsilon\nabla_k)\big) + W(\mathrm{i}\varepsilon\nabla_k) + \mathcal{O}(\varepsilon)$$

acting on some suitable space $\mathcal{H}_{\mathrm{eff}}$ of functions of k.

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For the case $A_0 = 0$ this was achieved in *Panati, Spohn, T.* '03, where we also computed the first order correction term to Peierls substitution. In this case

$$\mathcal{H}_{\mathrm{eff}} = L^2(\mathbb{T}_k^*)$$

where $\mathbb{T}_k^*$ denotes M^* with opposing edges identified.

4. Results

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Definition Let the bundle $\pi : \Xi_\tau \to \mathbb{T}^*$ with typical fibre $\mathcal{H}_\mathrm{f}$ be given by

$$\Xi_\tau := (\mathbb{R}^2 \times \mathcal{H}_\mathrm{f})/\sim_\tau ,$$

where

$$(k, \varphi) \sim_\tau (k', \varphi') \quad :\Leftrightarrow \quad \exists \gamma^* \in \Gamma^* : \; k' = k - \gamma^* \text{ and } \varphi' = \tau(\gamma^*)\varphi \, .$$

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As a consequence, $\mathcal{H}_\tau = L^2(\Xi_\tau)$.

The **Bloch bundle** Ξ_n associated to the isolated Bloch band $E_n(k)$ is the subbundle of Ξ_τ given by

$$\Xi_n := \{(k,\varphi) \in \mathbb{R}^2 \times \mathcal{H}_\mathrm{f} \,|\, \varphi \in P_n(k)\mathcal{H}_\mathrm{f}\}/\!\sim_\tau\,.$$

4. Results

For $A_0 = 0$ it was shown by **Panati** '07 that Bloch bundles are always trivializable. On the other hand, for $A_0 \neq 0$ Bloch bundles are non-trivial in general, as they have non-zero Chern numbers.

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Theorem (Freund, T. '13)

To each isolated magnetic Bloch band E_n there exists a unitary

$$U_n^\varepsilon : \mathrm{ran}\Pi_n^\varepsilon \to \mathcal{H}_\theta$$

such that $H_n^{\mathrm{eff}} := U_n^\varepsilon \Pi_n^\varepsilon H^\varepsilon \Pi_n^\varepsilon U_n^{\varepsilon *}$ satisfies

$$H_n^{\mathrm{eff}} = E_n\big(k + A(\mathrm{i}\varepsilon\nabla_k^\theta)\big) + W(\mathrm{i}\varepsilon\nabla_k^\theta) + \mathcal{O}(\varepsilon)\,.$$

Here $\mathcal{H}_\theta = L^2(\Xi_\theta)$ contains L^2-section of a line-bundle Ξ_θ over the torus $\mathbb{T}^*$ with connection ∇^θ determined by the Chern number $\theta \in \mathbb{Z}$ of the Bloch bundle Ξ_n.

4. Results

Main steps in the construction:

- Construct geometric Weyl-calculus for pseudodifferential operators acting on sections of non-trivial vector bundles over the torus.

  Based on *Widom '80, Safarov '98, Pflaum '98, Sharafutdinov '05, Hansen '10.*

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- Compute asymptotic expansion of H_n^{eff}.

5. Canonical models for non-zero Chern numbers

The dispersion of the discrete Laplacian on $\mathbb{Z}^2$ is

$$E(k) = 2\big(\cos(k_1) + \cos(k_2)\big) = \mathrm{e}^{\mathrm{i}k_1} + \mathrm{e}^{-\mathrm{i}k_1} + \mathrm{e}^{\mathrm{i}k_2} + \mathrm{e}^{-\mathrm{i}k_2}\,.$$

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The Fourier transform of the **discrete magnetic Laplacian** is

$$H^B_{\mathrm{Hof}} = \mathrm{e}^{\mathrm{i}\mathcal{K}_1} + \mathrm{e}^{-\mathrm{i}\mathcal{K}_1} + \mathrm{e}^{\mathrm{i}\mathcal{K}_2} + \mathrm{e}^{-\mathrm{i}\mathcal{K}_2}\,,$$

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H^B_{Hof} is called the **Hofstadter Hamiltonian** and it is given exactly by Peierls substitution,

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The Hofstadter Hamiltonian is the **canonical model for a non-magnetic Bloch band** perturbed by a small magnetic field B.

5. Canonical models for non-zero Chern numbers

Taking $E(k)$ as the dispersion of a magnetic Bloch band with Chern number $\theta \in \mathbb{Z}$, our Peierls substitution yields the **canoncial model for a magnetic Bloch band** perturbed by a small magnetic field B.

$$H_\theta^B := E\big(k - A(\mathrm{i}\nabla_k^\theta)\big) = \mathrm{e}^{\mathrm{i}\mathcal{K}_1^\theta} + \mathrm{e}^{-\mathrm{i}\mathcal{K}_1^\theta} + \mathrm{e}^{\mathrm{i}\mathcal{K}_2^\theta} + \mathrm{e}^{-\mathrm{i}\mathcal{K}_2^\theta}$$

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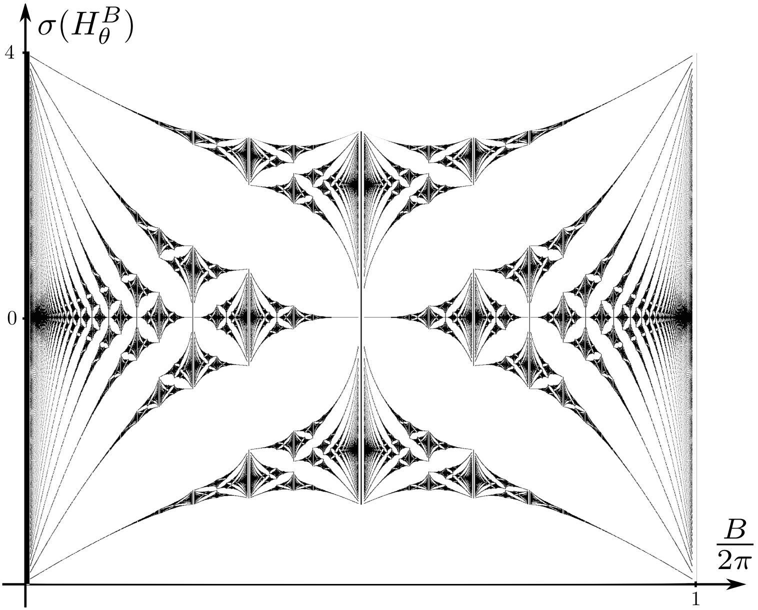
Note that $H_0^B \equiv H_{\mathrm{Hof}}^B$.

5. Colored butterflies of Osadchy and Avron



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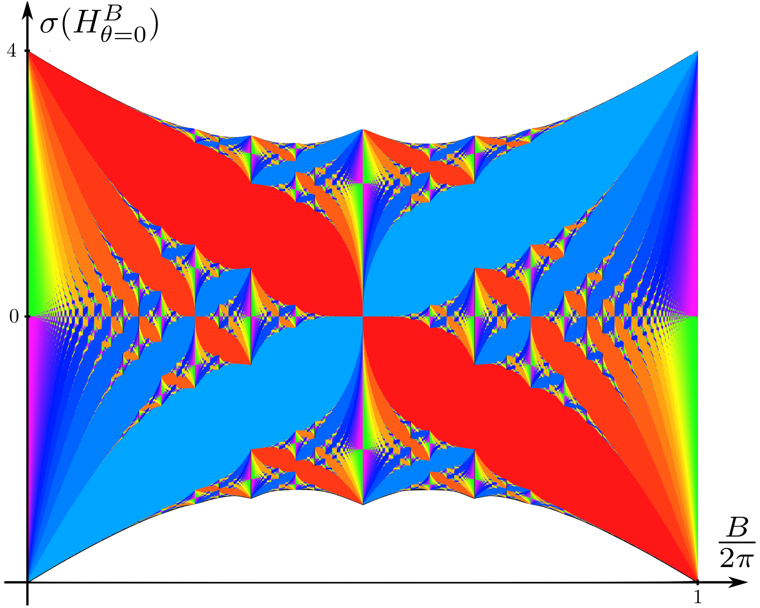

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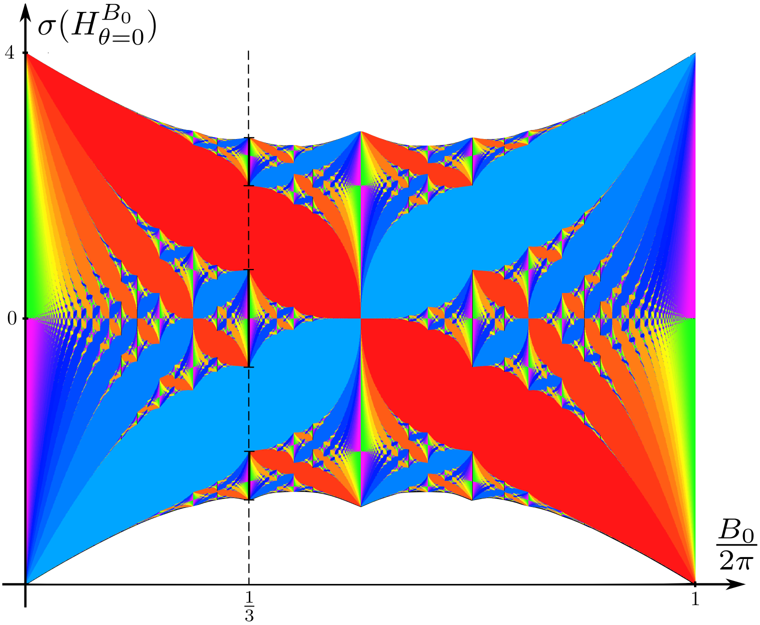

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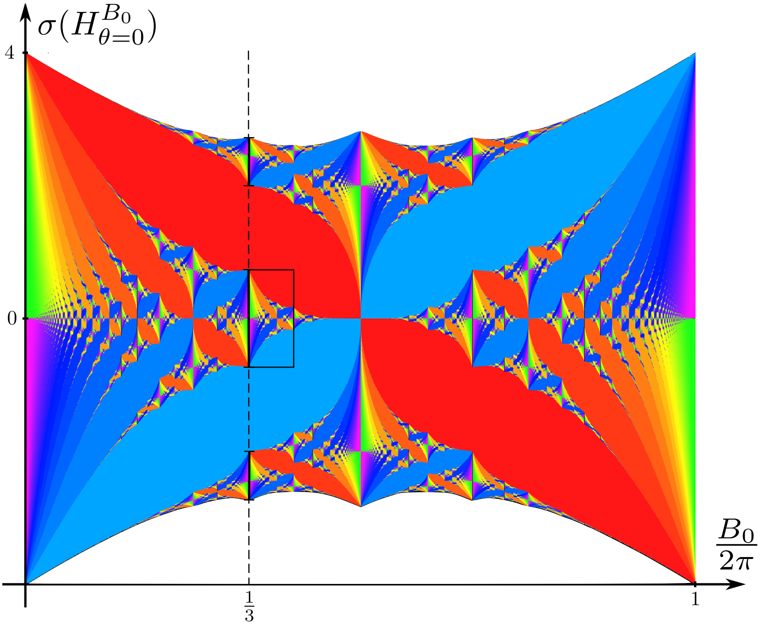

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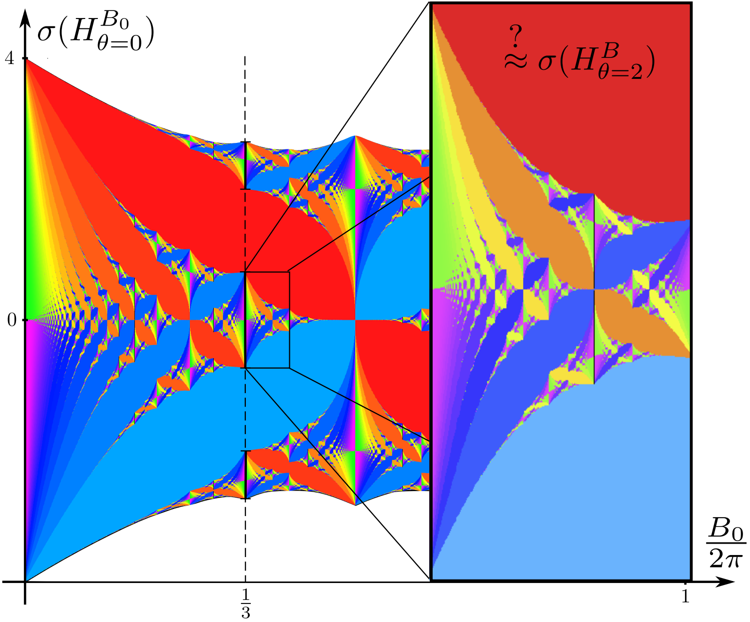

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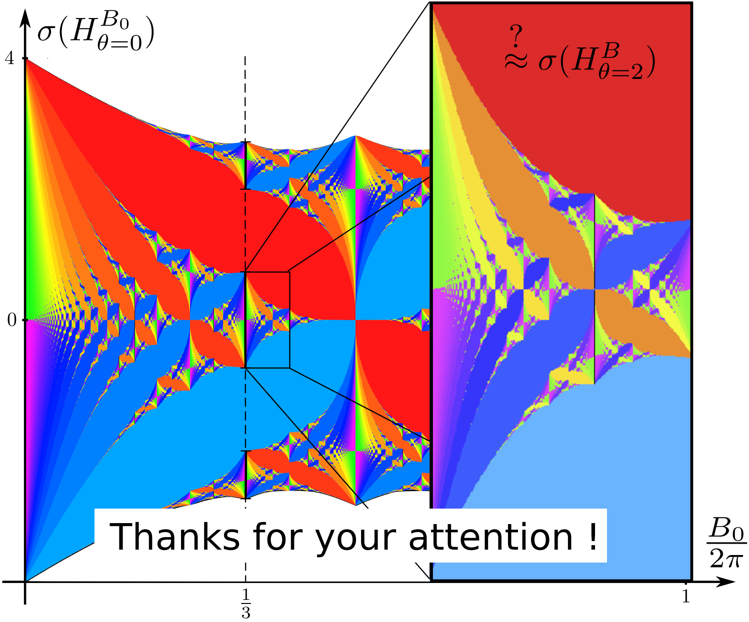

Thanks for your attention !